MATH 202A
APPLIED ALGEBRA I
FALL 2019

Homework week 8

Due by the beginning of class on Friday 22nd November (hand in via Gradescope).

For two norms $\|\cdot\|, \|\cdot\|'$ on the same vector space V, we write $\|\cdot\| \le \|\cdot\|'$ to mean $\|v\| \le \|v\|'$ for all $v \in V$.

Throughout, you may use any form on the Hahn–Banach theorem that has been stated in lectures (proven or otherwise).

1. For each of the following pairs of norms $\|\cdot\|, \|\cdot\|'$ on $\mathbb{C}^n$, find, with proof, the smallest constant $C > 0$ and the largest constant $c > 0$ such that $c\|v\| \le \|v\|' \le C\|v\|$. (These constants may depend on n.)

(a) $\|\cdot\| = \|\cdot\|_2$, $\|\cdot\|' = \|\cdot\|_\infty$.

(b) $\|\cdot\| = \|\cdot\|_2$, $\|\cdot\|' = \|\cdot\|_1$.

(c) $\|\cdot\| = \|\cdot\|_7$, $\|\cdot\|' = \|\cdot\|_{17}$.

[If you prefer to prove a general result, feel free.]

2. Let V be a finite-dimensional vector space.

(a) Suppose $\|\cdot\|, \|\cdot\|'$ are two norms on V such that $\|\cdot\| \le \|\cdot\|'$. Prove that $\|\cdot\|'^* \le \|\cdot\|^*$.

(b) If $\|\cdot\|$ is a norm on V and $\alpha \in \mathbb{R}_{>0}$ is a scalar, write $\alpha\|\cdot\|$ for the norm $v \mapsto \alpha\|v\|$. Prove that the dual norm to $\alpha\|\cdot\|$ is $(1/\alpha)\|\cdot\|^*$.

3. Let V, W be finite-dimensional normed spaces, with the norms denoted $\|\cdot\|_V$, $\|\cdot\|_W$. Consider V^*, W^* as normed spaces with the norms $\|\cdot\|_V^*$, $\|\cdot\|_W^*$ respectively.

Also, suppose $\phi\colon V \to W$ is a linear map.

(a) Prove that $\|\phi^*\|_{\text{op}} \le \|\phi\|_{\text{op}}$.

(b) Prove that $\|\phi^*\|_{\text{op}} = \|\phi\|_{\text{op}}$.

[**Hint**: stating part (a) before part (b) is a hint.]

(c) Reflect on the relationship between your proof of Pset 4 Q3 on the one hand, and (a) here together with Example 5.3.1 on the other hand. [You do not need to write anything for this part.]

4. Let V be a finite-dimensional vector space and $\|\cdot\|, \|\cdot\|'$ two norms on V. Consider the following functions $V \to \mathbb{R}$:—

$$
\begin{aligned}
v &\mapsto \|v\| + \|v\|' \\
v &\mapsto \max(\|v\|, \|v\|') \\
v &\mapsto \inf_{\substack{x,y \in V \\ x+y=v}} \left(\|x\| + \|y\|'\right) \\
v &\mapsto \inf_{\substack{x,y \in V \\ x+y=v}} \max(\|x\|, \|y\|').
\end{aligned}
$$

We denote there respectively by $\|\cdot\|+\|\cdot\|'$, $\max(\|\cdot\|, \|\cdot\|')$, $\operatorname{coplus}(\|\cdot\|, \|\cdot\|')$ and $\operatorname{comax}(\|\cdot\|, \|\cdot\|')$. Note these last two are non-standard notation.

(a) Very briefly, verify that these are all norms on V.

(b) Prove that $\big(\|\cdot\| + \|\cdot\|'\big)^* \leq \text{comax}(\|\cdot\|^*, \|\cdot\|'^*)$.

(c) Prove that $\big(\max(\|\cdot\|, \|\cdot\|')\big)^* \leq \text{coplus}(\|\cdot\|^*, \|\cdot\|'^*))$.

[Note: these proofs may have many common steps. You should feel free to merge you answers or otherwise avoid repeating yourself too much.]

(d') Prove that $\big(\text{comax}(\|\cdot\|, \|\cdot\|')\big)^* \geq \|\cdot\|^* + \|\cdot\|'^*$.

[**Hint:** Given $\phi \in V^*$, by definition you are trying to find $v \in V$ such that $|\phi(v)|$ is large but $\|v\|_{\text{comax}}$ is small. Note this is the same as trying to find $x, y \in V$ with $|\phi(x+y)|$ large but $\max(\|x\|, \|y\|')$ small. You can do this directly from the definitions of $\|\phi\|^*$ and $\|\phi\|'^*$.]

(e') Prove that $\big(\text{coplus}(\|\cdot\|, \|\cdot\|')\big)^* \geq \max(\|\cdot\|^*, \|\cdot\|'^*)$.

[**Hint:** see (d').]

(f) Prove that in fact equality holds in (b),(c),(d'),(e').

[**Hint:** now you shouldn't have to do more work.]

5. Consider the vector space $\mathbb{C}^n$ with the usual ℓ^p-norms.

(a) Prove that $\|\cdot\|_2 \leq \frac{1}{2}\|\cdot\|_{4/3} + \frac{1}{2}\|\cdot\|_4$.

[Here we use the notation developed in previous questions.]

(b) Let $x \in \mathbb{C}^n$ be a vector with $\|x\|_2 \leq 1$. Prove that there is a decomposition $x = y + z$ where $y, z \in \mathbb{C}^n$ satisfy $\|y\|_{4/3} \leq 1/2$ and $\|z\|_4 \leq 1/2$.

[**Note**: you could give a direct proof of this statement, i.e. by finding an explicit formula for y_i and z_i, using calculus or otherwise. This is strongly discouraged. You should attempt to obtain this fact "for free" from (a), by leveraging the results from the rest of the pset.]