MATH 202A
APPLIED ALGEBRA I
FALL 2019

Homework week 7

Due by the beginning of class on Friday 15th November (hand in via Gradescope).

1. Suppose V is a finite-dimensional vector space, $\phi\colon V \to V$ is a nilpotent linear map, $m \geq 1$, $n_1, \dots, n_m \geq 1$ are integers ($n_1 + \dots + n_m = \dim V$) and $v_{i,j}$ ($1 \leq i \leq m$, $1 \leq j \leq n_i$) are some basis for V such that

$$\phi(v_{i,j}) = \begin{cases} v_{i,j-1} & : j > 1 \\ 0 & : j = 1 \end{cases}$$

as guaranteed by Theorem 4.2.7.

(a) Prove that

$$\dim \ker(\phi^k) = \sum_{i=1}^{m} \min(k, n_i).$$

(b) Hence, for each integer $r \geq 1$, find an expression for $|\{i\colon 1 \leq i \leq m,\ n_i = r\}|$ (the number of blocks of size r) in terms of the quantities $\dim \ker(\phi^k)$.

[In other words, this shows nilpotent normal form is unique up to permuting the blocks.]

2. Consider the space V of complex sequences $(a_0, a_1, a_2, \dots)$ satisfying

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \dots + r_d a_{n-d}$$

for all $n \geq d$, where $d \geq 1$ is an integer and $r_1, \dots, r_d \in \mathbb{C}$ are fixed constants.

Show that there is a basis for V consisting of sequences of the form

$$a_n = \lambda^n \binom{n}{\ell}$$

for some $\lambda \in \mathbb{C}$ and $\ell \geq 0$.

[**Hint**: let $\phi\colon V \to V$ denote the (infinite) left shift, as in Pset 6 Q6. One approach is to consider $v = (a_0, a_1, \dots) \in G(\lambda, \phi)$ and find an explicit formula for a_n.]

3.(a) Suppose V is an inner product space with inner product $\langle -, - \rangle$ and associated norm $\| \cdot \|$. Prove that the *parallelogram law* holds for $\| \cdot \|$: that is, for all $v, w \in V$ we have

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

(b) For any $n \geq 1$ and $1 \leq p \leq \infty$, $p \neq 2$, find vectors $v, w \in \mathbb{R}^n$ such that

$$\|v + w\|_p^2 + \|v - w\|_p^2 \neq 2\|v\|_p^2 + 2\|w\|_p^2.$$

4.(a) Suppose V is a finite-dimensional real vector space and $\| \cdot \|$ is a norm on V obeying the parallelogram law: that is,

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

for all $v, w \in V$. Prove that there is an inner product $\langle -, - \rangle$ on V such that $\|v\| = \sqrt{\langle v, v \rangle}$.

[**Hint**: consider the quantity $\frac{1}{4}\left(\|v + w\|^2 - \|v - w\|^2\right)$.]

(b) Prove the same statement as (i) assuming now that V is a complex vector space.

[**Hint**: it may now help to build an expression from the quantities $\|v \pm w\|^2$ and $\|v \pm iw\|^2$.]

5. Let $V = \mathbb{R}^n$, and identify $V^* = \mathbb{R}^n$ in the usual way; i.e., a vector $(a_1, \ldots, a_n) \in \mathbb{R}^n$ corresponds to a linear map $\mathbb{R}^n \to \mathbb{R}$,

$$(x_1, \ldots, x_n) \mapsto a_1 x_1 + \cdots + a_n x_n.$$

For each of the norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ on $\mathbb{R}^n$, determine carefully the value of the corresponding dual norms $\|(a_1, \ldots, a_n)\|_\infty^*$, $\|(a_1, \ldots, a_n)\|_1^*$ for all $(a_1, \ldots, a_n) \in \mathbb{R}^n \cong V^*$.