MATH 202A
APPLIED ALGEBRA I
FALL 2019

Homework week 6

Due by the beginning of class on Friday 8th November (hand in via Gradescope).

1. Let V be a finite-dimensional vector space over a field F and $\phi\colon V \to V$ a linear map.

(a) Prove that if ϕ has an eigenvalue $\lambda \neq 0$ then ϕ is not nilpotent.

(b) Prove that, if $F = \mathbb{C}$ and $\operatorname{spec}(\phi) = \{0\}$ then ϕ is nilpotent. [For this question, do not use Jordan Normal Form itself, though you may use facts that go into it.]

(c) If $F = \mathbb{R}$ and $V = \mathbb{R}^3$, find an example of a map ϕ such that $\operatorname{spec}(\phi) = \{0\}$ but ϕ is not nilpotent.

2. Let V be a finite-dimensional vector space and $\phi\colon V \to V$ a linear map.

Consider the subspaces

$$\{0\} \subseteq \ker \phi \subseteq \ker(\phi^2) \subseteq \dots$$

(a) Prove that if $\ker(\phi^k) = \ker(\phi^{k+1})$, then in fact $\ker \phi^\ell = \ker \phi^{\ell+1}$ for all $\ell \geq k$.

(b) Suppose ϕ is nilpotent. Prove that $\phi^r = 0$ holds for some $r \leq \dim V$. [Do not use the structure theorem of nilpotent maps to answer this part.]

3. Let V be a finite dimensional vector space and $\phi\colon V \to V$ a linear map. Suppose $v \in V$, $v \neq 0$, and suppose that for some non-negative integer $m \geq 0$ we have

$$\phi^m(v) \neq 0$$

but

$$\phi^{m+1}(v) = 0 \ .$$

Show that $v, \phi(v), \dots, \phi^m(v)$ are linearly independent.

4. Let V be a finite dimensional vector space, and $\phi, \psi\colon V \to V$ linear maps. Show that if $\phi \circ \psi$ is nilpotent then so is $\psi \circ \phi$.

5. (a) Let V be finite dimensional and let $\psi\colon V \to V$ be a nilpotent map, where $\psi^k = 0$. Write $\phi = \psi + \lambda \operatorname{id}$ for some $\lambda \in F$. Show that for any integer $m \geq 0$,

$$\phi^m = \sum_{r=0}^{\max(k-1,m)} \lambda^{m-r} \binom{m}{r} \psi^r$$

(i.e., when $m \geq k-1$ this is

$$\lambda^m \operatorname{id} + \binom{m}{1} \lambda^{m-1} \psi + \binom{m}{2} \lambda^{m-2} \psi^2 + \dots + \binom{m}{k-1} \lambda^{m-k+1} \psi^{k-1}$$

and for small m you should stop the sum at $\binom{m}{m} \lambda^0 \psi^m$.)

(b) For any integer $m \geq 0$, compute the matrix J_k^m. Using (a), or otherwise, also compute $J(k,\lambda)^m$. Here

$$J_k = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad J(k,\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}$$

are $k \times k$ matrices.

6. Consider the vector space of infinite complex sequences $(a_0, a_1, a_2, \dots)$. Let V denote the subspace that satisfy the homogeneous recursion relation

$$a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$$

for all $n \geq 3$. [You need not verify that this is a subspace.] Also let $\phi\colon V \to V$ denote the "infinite left shift" map

$$\phi(a_0, a_1, a_2, \dots) = (a_1, a_2, a_3, \dots).$$

You should satisfy yourself that this latter sequence is indeed in V, but do not need to write anything.

(a) Prove that the linear map $\psi\colon V \to \mathbb{C}^3$, $\psi(a_0, a_1, a_2, \dots) = (a_0, a_1, a_2)$ is an isomorphism, and write down a matrix for ϕ with respect to the basis $B = \psi^{-1}(e_1), \psi^{-1}(e_2), \psi^{-1}(e_3)$ where e_1, e_2, e_3 is the standard basis for $\mathbb{C}^3$.

(b) Hence, or otherwise, compute $\operatorname{spec}(\phi)$, and deduce that $\phi - 2\,\mathrm{id}_V$ is nilpotent.

(c) Using Q4(i), or otherwise, and noting that a_n is the first entry of $\phi^n(a_0, a_1, \dots)$, find a closed-form formula for a_n in terms of a_0, a_1, a_2.