MATH 202A
APPLIED ALGEBRA I
FALL 2019

Homework week 5

Due by the beginning of class on Friday 1st November (hand in via Gradescope).

Given an linear map ϕ between finite dimensional inner product spaces, we write $\sigma_1(\phi)$ for the largest singular value of ϕ.

1. If V is a finite-dimensional inner product space and $\phi\colon V \to V$ is a normal map with eigenvalues $\lambda_1, \dots, \lambda_n$. Prove that the singular values of ϕ are $|\lambda_1|, \dots, |\lambda_n|$.

2. Let U, V, W be finite-dimensional inner product spaces and $\phi\colon U \to V$, $\psi\colon V \to W$ linear maps, show that $\sigma_1(\psi \circ \phi) \le \sigma_1(\psi)\sigma_1(\phi)$.

3.(a) Let V be a finite dimensional inner product space and $\phi\colon V \to V$ a linear map. For any positive integer k, prove that $\sigma_1(\phi^k) \le \sigma_1(\phi)^k$. Prove that if ϕ is normal then $\sigma_1(\phi^k) = \sigma_1(\phi)^k$.

(b) Consider $V = \mathbb{C}^2$ and $\phi(x, y) = (x + (3/2)y, y)$. Prove that $\sigma_1(\phi)^{1000} = 2^{1000}$ but $\sigma_1(\phi^{1000}) \le 1500 + \frac{1}{1500}$.

4. Let V be an inner product space (not necessarily finite-dimensional) and $\phi\colon V \to V$ a self-adjoint, non-negative definite linear map (i.e., $\langle \phi(v), w\rangle = \langle v, \phi(w)\rangle$ for all $v, w \in V$, and $\langle \phi(v), v\rangle \ge 0$ for all $v \in V$).

(a) Prove carefully that for all $v, w \in V$, $|\langle \phi(v), w\rangle|^2 \le \langle \phi(v), v\rangle\langle \phi(w), w\rangle$.

(b) Prove that if $\langle \phi(v), v\rangle = 0$ then $\phi(v) = 0$. [If you proved Pset 4 Q1(a) like this already, great; if not this gives a new proof.]

(c) Now suppose $\psi\colon V \to V$ is another self-adjoint, non-negative definite linear map and $\langle (\phi + \psi)(v), v\rangle = 0$. Prove that $\phi(v) = \psi(v) = 0$.

5. Let V be an inner product space (not necessarily finite-dimensional) and let $\phi, \psi\colon V \to V$ be two self-adjoint, non-negative definite linear maps such that $\phi^2 = \psi^2$. Suppose moreover that $\phi \circ \psi = \psi \circ \phi$. Prove that $\phi = \psi$.

[**Hint:** You could look at something like $\langle (\phi + \psi)(\phi - \psi)(v), (\phi - \psi)(v)\rangle$ for arbitrary $v \in V$, and remember Q1. But there are probably many other starting points.]

6. My office has three chairs, arranged in a row.[1] Every hour, I receive an instruction by email from the university authorities. If the instruction says "FLIP", I roll a fair 2-sided die and accordingly move to one of the chairs I am not currently sitting in, with probability 1/2 each.

If the instruction says "FLOP", I roll a fair 4-sided die and move chairs, or not, based on the rule given in Pset 3 Q7.

So, if my probabilities of being in each chair one hour are (p_L, p_M, p_R), then the probabilities for the next hour are given by

$$\psi(p_L, p_M, p_R) = \left(\frac{1}{2}p_M + \frac{1}{2}p_R, \frac{1}{2}p_L + \frac{1}{2}p_R, \frac{1}{2}p_L + \frac{1}{2}p_M\right)$$

[1] My office does actually now have chairs in. They are not arranged in a row; that bit is a lie.

if the instruction is "FLIP", and

$$\phi(p_L, p_M, p_R) = \left(\frac{3}{4}p_L + \frac{1}{4}p_M, \frac{1}{4}p_L + \frac{1}{2}p_M + \frac{1}{4}p_R, \frac{1}{4}p_M + \frac{3}{4}p_R\right)$$

if the instruction is "FLOP".

The university authorities are a law unto themselves, and their instructions are unpredictable, arbitrary and follow no predictable pattern. In particular you may *not* assume that FLIP and FLOP occur according to any particular random rule.

I start in the left chair. Show that the probability I am in the left chair again after 1000 straight hours in my office is $1/3 + \varepsilon$ where $|\varepsilon| \leq 2 \cdot (3/4)^{1000}$ (which is around $2.3 \cdot 10^{-125}$).

[**Hint:** it might help to consider Q1, Q2 and the subspace $\{1, 1, 1\}^{\perp}$.]