MATH 202A
APPLIED ALGEBRA I
FALL 2019

HOMEWORK WEEK 10

Due by the beginning of class on Friday 6th December (hand in via Gradescope).

1. Let U, V be finite-dimensional inner product spaces, and $\phi \in \mathcal{L}(U, V)$ a linear map. Moreover, let $u_1, \dots, u_n$ be an orthonormal basis for U, $v_1, \dots, v_m$ an orthonormal basis for V, and $\sigma_1, \dots, \sigma_n \geq 0$ be the singular values of ϕ (where $n = \dim U$, $m = \dim V$), such that

$$\phi(u_i) = \begin{cases} \sigma_i v_i & i \leq \min(n, m) \\ 0 & \text{otherwise.} \end{cases}$$

Now let $\Phi \colon U \oplus V \to U \oplus V$ be the linear map $\Phi(u, v) = \big(\phi^*(v), \phi(u)\big)$.

(a) Assuming $U \oplus V$ is given the structure of an inner product space, with $\langle (u, v), (u', v') \rangle = \langle u, u' \rangle_U + \langle v, v' \rangle_V$, prove that Φ is self-adjoint.

(b) Compute the eigenvalues, and a corresponding orthonormal basis of eigenvectors, of Φ, in terms of the data σ_i, u_i, v_i.

[This is a useful trick for using facts about eigenvalues to deduce ones about singular values.]

2. Let $U = \mathbb{C}^n$, $V = \mathbb{C}^m$ with the usual dot product, and $\phi \in \mathcal{L}(U, V)$ a linear map with matrix A with respect to the standard bases. Let $k \geq 1$ be an integer.

Prove that the Schatten $2k$-norm of ϕ is given by

$$\|\phi\|_{S(2k)}^{2k} = \operatorname{tr}\big((\phi^* \circ \phi)^k\big) = \sum_{i_1, \dots, i_k = 1}^{m} \sum_{j_1, \dots, j_k = 1}^{n} \prod_{r=1}^{k} A_{i_r j_r} \overline{A_{i_{r+1} j_r}}$$

where we adopt the convention that $i_{k+1} = i_1$.

3. Let U, V be finite-dimensional inner product spaces, and $\phi \colon U \to V$ a linear map with singular values $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$, where $n = \dim U$. Let k, $1 \leq k \leq n$, be an integer.

(a) Prove that

$$\sup_{\substack{\psi \in \mathcal{L}(U,V) \\ \|\psi\|_{\mathrm{op}} \leq 1 \\ \operatorname{rank}(\psi) \leq k}} |\langle \phi, \psi \rangle_{\mathrm{Frob}}| = \sum_{i=1}^{k} \sigma_i.$$

You may use without proof the following fact: if $a_1 \geq \cdots \geq a_n \geq 0$ are real numbers and $b_1, \dots, b_n \geq 0$ satisfy $b_i \leq 1$ and $\sum_{i=1}^n b_i = k$, then

$$\sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{k} a_i;$$

i.e., the maximum value of the left-hand side under these hypotheses is when $b_1 = b_2 = \cdots = b_k = 1$ and $b_{k+1} = \cdots = b_n = 0$.

[**Hint**: the proof of Hölder's inequality for Schatten norms may be useful here.]

(b) Deduce that

$$\|\phi\| = \sum_{i=1}^{k} \sigma_i$$

defines a norm on $\mathcal{L}(U, V)$ (called the k-th *Ky–Fan norm*).