MATH 202A
APPLIED ALGEBRA I
FALL 2019

Homework week 1

Due by the beginning of class on Friday 4th October (hand in via Gradescope).

1. Prove, using only the results from Section 1.2, that (i) any finite-dimensional vector space has a basis, and (ii) any two bases of a finite dimensional vector space V have the same size.

[Recall that we say a vector spaces is finite-dimensional to mean that it has a finite spanning set.]

2. Let V be a finite dimensional vector space over the finite field $\mathbb{F}_p$ (p prime). Show that $|V|$, the number of distinct vectors in V, is finite and a power of p.

3. Let V, W be vector spaces, $U \subseteq V$ a subspace, and $\phi\colon V \to W$ a linear map. Suppose also that $\ker \phi \supseteq U$.

Prove that there exists a unique linear map $\psi\colon V/U \to W$ such that $\phi(v) = \psi(v + U)$ for all $v \in V$.

Prove that ψ is injective if and only if $U = \ker \phi$.

4. Let V be a finite-dimensional vector space and $U \subseteq V$ a subspace. Suppose that $u_1, \dots, u_k$ are a basis for U, and $v_1, \dots, v_m$ are vectors in V such that $v_1 + U, \dots, v_m + U$ is a basis for V/U.

Prove that $u_1, \dots, u_k, v_1, \dots, v_m$ is a basis for V (and hence $\dim V = \dim U + \dim(V/U)$).

5. Suppose V is a vector space over $\mathbb{R}$, $w_1, w_2, w_3 \in V$ span V and u_1, u_2, u_3 are linearly independent where

$$\begin{aligned} u_1 &= 4w_1 - 3w_3 \\ u_2 &= -w_1 + w_3 \\ u_3 &= w_1 + w_2 + w_3\,. \end{aligned}$$

By Theorem 1.2.1, we can add $3 - 3 = 0$ vectors from w_1, w_2, w_3 to u_1, u_2, u_3 to make a spanning set; i.e. u_1, u_2, u_3 already spans V. In particular there exist coefficients $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_1u_1 + a_2u_2 + a_3u_3 = w_3$.

Our proofs are constructive. So, we can (and will) run the proof of Theorem 1.2.1 for $k = m = 3$ and these u_i, w_i, line by line, to compute algorithmically what the scalars a_1, a_2, a_3 are. [For this question, don't compute them "by hand": follow the proof as directed.]

(a) Consider the list u_1, w_1, w_2, w_3. It is linearly dependent (by Lemma 1.2.3, as w_1, w_2, w_3 spans). Use the proof to find a dependence relation among these vectors. Hence (as in Lemma 1.2.2) find an expression for w_3 as a linear combination of u_1, w_1, w_2.

(b) Using (a), express u_2, u_3 as linear combinations of u_1, w_1, w_2.

(c) Consider the list u_2, u_1, w_1, w_2. It is linearly dependent (by Lemma 1.2.3, as u_1, w_1, w_2 spans). Use the proof, and (b), to find a dependence relation among these vectors. Hence (as in Lemma 1.2.2) find an expression for w_1 as a linear combination of u_1, u_2.

(d) Using (b) and (c), express u_3 as a linear combination of u_1, u_2, w_2.

(e) Consider the list u_3, u_2, u_1, w_2. It is linearly dependent (by Lemma 1.2.3). Use the proof, and (d), to find a dependence relation among these vectors. Hence (as in Lemma 1.2.2) find an expression for w_2 as a linear combination of u_1, u_2, u_3.

(f) Using your expressions for w_1 from (c), w_2 from (e), and w_3 from (a), write w_3 as a linear combination of u_1, u_2, u_3.

Note the similarity of this process with any other algorithms (Gaussian elimination / reducing to RREF / LU decomposition / etc.) with which you are familiar. [You need not write anything for this part.]