MATH 190A
WINTER 2021

Problem Set 6

Due by 2359 (11:59 PM) on Sunday February 14. Hand in via Gradescope.

You may discuss these problems among yourselves, but your final submitted solutions must be written by you alone, in your own words, without any other solution in front of you.

1. Let X and Y be topological spaces, and fix any point $x_0 \in X$. Prove that

$$\{x_0\} \times Y = \{(x, y) \in X \times Y : x = x_0\} \subseteq X \times Y$$

considered with the subspace topology from $X \times Y$ (which carries the product topology), is homeomorphic to Y.

2. Which of the following spaces are connected? In each case, give a full proof of your answer.

(a) The space $\{0, 1\}^{\mathbb{N}}$ with the topology from Pset 3 Q1.

(b) The subspace $X \subseteq \mathbb{R}^2$,

$$X = \big\{(x, y) \in \mathbb{R}^2 : \text{at least one of } x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\big\}.$$

(c) The subspace $Y \subseteq \mathbb{R}^2$,

$$Y = \big\{(x, y) \in \mathbb{R}^2 : x \notin \mathbb{Q} \text{ and } y \notin \mathbb{Q}\big\}.$$

3. Suppose $f : X \to Y$ is continuous and surjective, and X is path-connected. Prove that Y is path-connected.

4. Consider the space $X = \mathbb{R}^2 \setminus \{0\}$, with the subspace topology from $\mathbb{R}^2$.

(a) Prove that X is path-connected.

(b) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Prove that the images $f(X)$ and $f(\mathbb{R}^2)$ are connected.

(c) Prove that there is no continuous *injective* function $f : \mathbb{R}^2 \to \mathbb{R}$.

[In particuar, we have now shown that $\mathbb{R}^2$ and $\mathbb{R}$ are not homeomorphic!]

5. Suppose X, Y are topological spaces and $f : X \to Y$ is a function that is *locally constant*: that is, for every $x \in X$ there exists an open neighborhood $U \ni x$ such that $f(x) = f(y)$ for all $y \in U$ (i.e., f is constant on U).

(a) Prove that f is continuous.

(b) If X is connected, prove that f is constant.