MATH 190A
WINTER 2021

Problem Set 4

Due by 2359 (11:59 PM) on Sunday January 31. Hand in via Gradescope.

You may discuss these problems among yourselves, but your final submitted solutions must be written by you alone, in your own words, without any other solution in front of you.

1. Let $(X, \mathcal{U}_X)$ be a topological space and $Z \subseteq X$ a subspace with the subspace topology. For each statement, give either a proof or a counterexample.

(a) A subset $S \subseteq Z$ is closed in Z if and only if S is closed in X.

(b) A subset $S \subseteq Z$ is closed in Z if and only if there exists $T \subseteq X$ with $T \cap Z = S$.

(c) If $S \subseteq Z$ is closed in Z and $T \subseteq X$ satisfies $T \cap Z = S$ then T is closed in X.

2. Let X and Y be topological spaces, and write $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ for the projection maps.

(a) Prove that if $T \subseteq X$ and $\pi_1^{-1}(T)$ is open in $X \times Y$ then T is open.
[Remember this *doesn't* follow from the fact π_1 is continuous. You will have to delve more carefully into the definition of the product topology than that.]

(b) Give an example of X, Y and a set $S \subseteq X \times Y$ such that S is closed but $\pi_1(S)$ is not closed.

3. Let X and Y be two topological spaces. Let $x_1, x_2, \dots \in X$ and $y_1, y_2, \dots \in Y$ be two sequences.

(a) Let $(x, y) \in X \times Y$ be a point. Prove carefully that $(x_n, y_n) \to (x, y)$ if and only if $x_n \to x$ and $y_n \to y$.

(b) Let $S = \{x_1, x_2, \dots\}$, $T = \{y_1, y_2, \dots\}$ and $Z = \{(x_1, y_1), (x_2, y_2), \dots\}$. Suppose $x \in \overline{S}$ and $y \in \overline{T}$. Does it follow that $(x, y) \in \overline{Z}$? Give either a proof or a counterexample.