Let $(X,d)$ be a metric space. Given $x \in X$ and $r \geq 0$, consider the closed ball $B_r[x] = \{ y \in X : d(x,y) \leq r \}$.

(a) Prove that $B_r[x]$ is closed.

(b) Prove that $B_r[x]$ contains $\overline{B_r(x)}$, the closure of the open ball with the same radius and center.

(c) Is $B_r[x]$ always equal to the closure $\overline{B_r(x)}$? Justify your answer.

2. Give an example of a metric space $(X,d)$ and a point $x \in X$ such that the closed ball $B_{1.001}[x]$ (see Q1) contains 1000 disjoint closed balls of radius 1, $B_1[y_1], \ldots, B_1[y_{1000}] \subseteq B_{1.001}[x]$.

3. Let $(X,d_X)$ be a metric space and consider $X \times X$ as a metric space with the product metric $d_{X \times X}((x,y),(x',y')) = d_X(x,x') + d_X(y,y')$. Prove that the map

$$X \times X \to \mathbb{R}$$

$$(x,y) \mapsto d_X(x,y)$$

is continuous.