MATH 190A
WINTER 2021

Problem Set 1

Due by 2359 (11:59 PM) on Sunday January 10. Hand in via Gradescope.

You may discuss these problems among yourselves, but your final submitted solutions must be written by you alone, in your own words, without any other solution in front of you.

1. Let (X, d) be a metric space. Given $x \in X$ and $r \geq 0$, consider the *closed ball*

$$B_r[x] = \{y \in X : d(x, y) \leq r\}.$$

(a) Prove that $B_r[x]$ is closed.

(b) Prove that $B_r[x]$ contains $\overline{B_r(x)}$, the closure of the open ball with the same radius and center.

(c) Is $B_r[x]$ always equal to the closure $\overline{B_r(x)}$? Justify your answer.

2. Give an example of a metric space (X, d) and a point $x \in X$ such that the closed ball $B_{1.001}[x]$ (see Q1) contains 1000 disjoint closed balls of radius 1, $B_1[y_1], \ldots, B_1[y_{1000}] \subseteq B_{1.001}[x]$.

3. Let (X, d_X) be a metric space and consider $X \times X$ as a metric space with the product metric

$$d_{X\times X}\big((x, y), (x', y')\big) = d_X(x, x') + d_X(y, y').$$

Prove that the map

$$\begin{aligned} X \times X &\to \mathbb{R} \\ (x, y) &\mapsto d_X(x, y) \end{aligned}$$

is continuous.