MATH 142B
SPRING 2020
SECTION B00 (MANNERS)

HOMEWORK – WEEK 9

Due by 2359 (11:59 PM) on Tuesday June 2. Hand in via Gradescope.

You may discuss these problems among yourselves, but your final submitted solutions must be written by you alone. You may not receive direct assistance on these problems from the internet.

1. Let $f\colon [0,1] \to \mathbb{R}$ be Darboux integrable. Prove that for all $\varepsilon > 0$ there exists is a continuous function $g\colon [0,1] \to \mathbb{R}$ such that

$$\int_0^1 |f(x) - g(x)|\ dx \le \varepsilon.$$

[**Hint:** find a partition P with $U(f,P) - L(f,P)$ small and construct g out of P.]

2.(a) Let $f\colon [0,\infty) \to \mathbb{R}$ be a monotone decreasing function. Prove carefully that for integers $1 \le n < m$,

$$\int_{x=n}^{m} f(x)\ dx \le \sum_{k=n}^{m-1} f(k) \le \int_{x=n-1}^{m-1} f(x)\ dx.$$

[You may assume that f is Darboux integrable on any interval $[A,B]$ for $0 \le A < B$.]

(b) Prove carefully that the series

$$\sum_{n=10}^{\infty} \frac{1}{n \log n \log\log n}$$

diverges.
[**Hint:** consider the derivative of the function $g(x) = \log\log\log(x)$.]

3. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function, and for each $x \in [0,1]$ define $R_x\colon [0,1] \to \mathbb{R}$ by

$$R_x(t) = \begin{cases} x - t & : t \le x \\ 0 & : t \ge x. \end{cases}$$

Now let

$$g(x) = \int_0^1 f(t)\, R_x(t)\ dt.$$

Show that g is twice differentiable on $(0,1)$ and $g''(x) = f(x)$ for all $x \in (0,1)$.

[**Hint:** You can start by following the ideas from the proof of the Fundamental Theorem of Calculus II to analyze g'. What does the function $t \mapsto \frac{R_y(t) - R_x(t)}{y - x}$ look like?]