MATH 142B
SPRING 2020
SECTION B00 (MANNERS)

Homework – week 8

Due by 2359 (11:59 PM) on Tuesday May 26. Hand in via Gradescope.

You may discuss these problems among yourselves, but your final submitted solutions must be written by you alone. You may not receive direct assistance on these problems from the internet.

1. Prove carefully the following facts.

(a) If $f, g\colon [A,B] \to \mathbb{R}$ are (Darboux) integrable, and $a, b \in \mathbb{R}$ are real numbers, then

$$h(x) = af(x) + bg(x)$$

is Darboux integrable, and $\int_A^B h(x)\ dx = a\int_A^B f(x)\ dx + b\int_A^B g(x)\ dx$.

(b) If $f, g\colon [A,B] \to \mathbb{R}$ are (Darboux) integrable and $f(x) \le g(x)$ for all $x \in [A,B]$ then

$$\int_A^B f(x)\ dx \le \int_A^B g(x)\ dx.$$

(c) If $f\colon [A,B] \to \mathbb{R}$ is a bounded function, $C \in (A,B)$, and f is integrable on $[A,C]$ and on $[C,B]$, then f is integrable on $[A,B]$ and

$$\int_A^B f(x)\ dx = \int_A^C f(x)\ dx + \int_C^B f(x)\ dx.$$

2. Consider a bounded function $f\colon [0,1] \to \mathbb{R}$ such that f is discontinuous at $x = 1, 1/2, 1/3, 1/4, \ldots$ (i.e., at $x = 1/n$ for integers $n \ge 1$) but is continuous at every other $x \in [0,1]$.

(a) Prove that f is integrable.

(b) Give an example of such a function.

3. Suppose $f\colon [0,1] \to \mathbb{R}$ is integrable and satisfies $f(x) \ge 0$ for all $x \in [0,1]$. Moreover suppose $\int_0^1 f(x)\ dx = 0$.

(a) Prove the following statement. Given any values a, b with $0 \le a < b \le 1$, and given any $\varepsilon > 0$, there exist a', b' with $0 \le a \le a' < b' \le b \le 1$ such that $\sup_{x\in[a',b']} f(x) \le \varepsilon$.

(b) Prove that there is at least one value $x \in [0,1]$ with $f(x) = 0$.
[**Hint:** apply part (a) iteratively with $\varepsilon = 1/n$ for every positive integer n, to obtain nested intervals $[a_1,b_1] \supseteq [a_2,b_2] \supseteq [a_3,b_3] \supseteq \ldots$ with $\sup_{x\in[a_n,b_n]} f(x) \le 1/n$.]