MATH 142B
SPRING 2020
SECTION B00 (MANNERS)

HOMEWORK – WEEK 7

Due by 2359 (11:59 PM) on Tuesday May 19. Hand in via Gradescope.

You may discuss these problems among yourselves, but your final submitted solutions must be written by you alone. You may not receive direct assistance on these problems from the internet.

1. Consider the function

$$f\colon \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \begin{cases} e^{-1/x} & : x > 0 \\ 0 & : x \leq 0 \end{cases}.$$

(a) Prove that f is infinitely differentiable on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for all $k \geq 0$. [You do not need to compute an explicit formula for $f^{(k)}(x)$ for each k.]
[**Hint:** you may use without proof the following fact: for any $m \geq 0$,

$$\lim_{x \to 0^+} x^{-m} e^{-1/x} = 0.$$

Indeed, this is equivalent to

$$\lim_{y \to +\infty} y^m e^{-y} = 0$$

i.e. the assertion that e^y grows faster than any power of y as $y \to +\infty$.]

(b) Prove that the Taylor series for f converges for every $x \in \mathbb{R}$ but that it does not converge to $f(x)$ for any $x > 0$.

2. Suppose $f\colon [0,1] \to \mathbb{R}$ is a contiuous function, not the zero function, such that $f(x) \geq 0$ for all x. Prove that

$$\int_0^1 f(x)\ dx > 0.$$

[You may assume the statement that a continuous function $f\colon [0,1] \to \mathbb{R}$ is always Darboux integrable.]

3. Consider the following exotic function $f\colon [0,1] \to \mathbb{R}$. If $x \in [0,1]$ is rational, we write $x = a/b$ as a fraction in its lowest terms (i.e., a, b are positive coprime integers) and set $f(x) = 1/b$. If x is irrational, we set $f(x) = 0$.

Determine whether f is Darboux integrable. If you determine that it is, determine $\int_0^1 f(x)\ dx$.