MATH 142B
SPRING 2020
SECTION B00 (MANNERS)

Homework – week 6

Due by 2359 (11:59 PM) on Tuesday May 12. Hand in via Gradescope.

You may discuss these problems among yourselves, but your final submitted solutions must be written by you alone. You may not receive direct assistance on these problems from the internet.

1. For this question, you may assume the following form of L'Hopital's rule: if $f, g\colon [0,1) \to \mathbb{R}$ are two continuous functions that are differentiable on $(0,1)$, $f(0) = g(0) = 0$, $g'(x) \neq 0$ for $x \in (0,1)$ and

$$\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = L.$$

[You may not use more general forms of L'Hopital that trivialize the question.]

Suppose now that $F, G\colon [1,\infty) \to \mathbb{R}$ are two continuous functions that are differentiable on $(1,\infty)$, which have $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} G(x) = 0$, $G'(x) \neq 0$ for $x \in (1,\infty)$, and

$$\lim_{x\to\infty} \frac{F'(x)}{G'(x)} = L.$$

Define $f, g\colon [0,1] \to \mathbb{R}$ by $f(x) = F(1/x)$ and $g(x) = G(1/x)$ where $x > 0$, and $f(0) = g(0) = 0$.

(a) Prove carefully that f, g are continuous.

(b) Prove that f and g are differentiable on $(0,1)$ and compute the derivatives f', g'.

(c) Prove carefully that $\lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = L$.

(d) Prove carefully that $\lim_{x\to\infty} \frac{F(x)}{G(x)} = L$.

2. Consider two functions

$$f, g\colon (-1,1) \to \mathbb{R}$$
$$f(x) = \begin{cases} x - x^2 \sin(1/x)\cos(1/x) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$
$$g(x) = \begin{cases} \big(x - x^2 \sin(1/x)\cos(1/x)\big)e^{\sin(1/x)} & : x \neq 0 \\ 0 & : x = 0 \end{cases}.$$

You may use without proof that f, g are continuous at 0.

(a) Show that for $x \in (-1, 1)$, $x \neq 0$:

$$f'(x) = 2\cos(1/x)^2 - 2x\sin(1/x)\cos(1/x)$$

$$g'(x) = \left(2\cos(1/x)^2 - 2x\sin(1/x)\cos(1/x) - \frac{1}{x}\cos(1/x) + \sin(1/x)\cos(1/x)^2\right)e^{\sin(1/x)}.$$

(b) Prove that

$$\lim_{x\to 0} \frac{2\cos(1/x) - 2x\sin(1/x)}{\big(2\cos(1/x) - 2x\sin(1/x) - \frac{1}{x} + \sin(1/x)\cos(1/x)\big)\, e^{\sin(1/x)}} = 0.$$

(c) Prove that the limit $\lim_{x\to 0} \frac{f(x)}{g(x)}$ does not exist.

(d) Why does this not contradict L'Hopital's rule?

3. Consider $f(x) = \sqrt{1+x}$.

(a) Prove that when $x < 0$,

$$0 \geq f(x) - \big(1 + x/2 - x^2/8\big) \geq x^3/16.$$

[Note RHS is negative.] [**Hint:** use Taylor's theorem.]

(b) Using $x = -0.02$, find an approximation to $\sqrt{2}$ and prove that it is correct to 5 decimal places.