**MATH 142B**
**SPRING 2020**
**SECTION B00 (MANNERS)**

# Homework – week 4

Due by 2359 (11:59 PM) on Tuesday April 28. Hand in via Gradescope.

You may discuss these problems among yourselves, but your final submitted solutions must be written by you alone. You may not receive direct assistance on these problems from the internet.

**1.** Prove that there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ with the following properties:—

(i) $f(0) = 0$, $f$ is differentiable at 0, and $f'(0) = 1$;
(ii) for every $x \in \mathbb{R}$, the value $f(x)$ is one of $\{0, \pm 1, \pm 1/2, \pm 1/3, \dots\}$.)

[**Hint:** you may use without proof the fact that if $f\colon \mathbb{R} \to \mathbb{R}$ is a function such that

$$x - x^2 \leq f(x) \leq x + x^2$$

for all $x \in \mathbb{R}$, then $f$ is differentiable at 0 and $f'(0) = 1$.]

*This is supposed to be surprising. For example, it is not true that if $f'(a) \neq 0$ then $f$ is bijective on some small interval around $a$.*

**2.** This question considers two variants of the Chain Rule, one of which is true and one of which is false.

Suppose that $f, g\colon \mathbb{R} \to \mathbb{R}$ are two functions, and $a \in \mathbb{R}$ is some point.

**(a)** Suppose that (i) $(g \circ f)$ is differentiable at $a$, (ii) $g$ is differentiable at $f(a)$, (iii) $g'(f(a)) \neq 0$, and (iv) $f$ is continuous at $a$. Prove that $f$ is differentiable at $a$, and that

$$f'(a) = \frac{(g \circ f)'(a)}{g'(f(a))}.$$

[**Hint:** Follow the proof of the chain rule very closely, and recall that if $G$ is continuous at $y$ and $G(y) \neq 0$ then $(1/G)$ is continuous at $y$ also. If you do not use all properties (i)–(iv) in a meaningful way, your answer is probably wrong.]

**(b)** Now instead suppose that (i) $(g \circ f)$ is differentiable at $a$, (ii) $f$ is differentiable at $a$, (iii) $f'(a) \neq 0$, and (iv) $g$ is continuous at $f(a)$. Give an example to show that $g$ need not necessarily be differentiable at $f(a)$.
[**Hint**: if you like, you can take $a = 0$,

$$g(x) = \begin{cases} x \sin(\pi/x) & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

and $f$ to be a function satisfying the conditions in Q1.]