MATH 142B
SPRING 2020
SECTION B00 (MANNERS)

Homework – week 10

These problems are optional and will not be graded. Do not turn them in.

1. In this question, we use our knowledge of differentiating power series to deduce some further properties of the log and exp functions.

We define functions

$$f\colon \mathbb{R} \to \mathbb{R}$$
$$x \mapsto \sum_{n\geq 0} x^n/n!$$

and a function $g\colon (-1,1) \to \mathbb{R}$ by

$$g(x) = \sum_{n\geq 1} (-1)^{n-1} x^n/n = x - x^2/2 + x^3/3 - x^4/4 + \ldots;$$

finally, let

$$h(x) = \begin{cases} g(x-1) & : 0 < x < 2 \\ -g(1/x-1) & : x > 1/2 \end{cases}.$$

Note that it is not currently obvious that h is well-defined. You may assume the properties of $f(x)$ given in Proposition 6.3.1. [Secretly $f(x) = e^x$ and $g(x) = \log(1+x)$, but we don't know that yet.]

(a) Prove that g is differentiable on $(-1,1)$ and $g'(x) = \frac{1}{1+x}$ for all $x \in (-1,1)$.

(b) For $x \in (1/2, 2)$ let $G(x) = g(x-1) + g(1/x-1)$. Prove that $G(x) = 0$ for all $x \in (1/2,2)$, and hence that h is well-defined. [See the proof of Proposition 6.3.1 for pointers.]

(c) Let $R(x) = g(f(x)-1)$. Prove carefully that $R(x) = x$ for all $x \in \mathbb{R}$. [**Hint:** consider $R'(x)$.]

2. An *improper integral* is when we take a limit of some integral $\int_a^b f(x)\,dx$ as a or b approach some value. This allows us to write down intergals that might otherwise fail to exist.

In each of the following cases, either compute the value of the improper integral or show that it is undefined.

(a) $\lim_{b\to+\infty} \int_1^b (1/x^2)\,dx$ (usually denoted $\int_1^\infty \ldots$).

(b) $\lim_{b\to+\infty} \int_0^b \frac{x}{1+x^2}\,dx$.

(c) $\lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt[3]{x}}\,dx$ (you could write this as $\int_{0^+}^1 \ldots$).

(d) $\lim_{a\to 0^+}\left[\int_{-1}^{-a}\frac{1}{\sqrt[3]{x}}\,dx+\int_a^1\frac{1}{\sqrt[3]{x}}\,dx\right]$.

(e) $\lim_{b\to+\infty}\int_{-b}^{b}\frac{x}{1+x^2}\,dx$ (sometimes called the "Cauchy principal value").

3. Let $f\colon \mathbb{R}\to\mathbb{R}$ be a differentiable function such that $f'\colon \mathbb{R}\to\mathbb{R}$ is continuous, and let $g\colon [0,1]\to\mathbb{R}$ be an integrable function on $[0,1]$.

(a) Let $a_n\to 0$ be a sequence of real numbers and let $[A,B]$ be a closed interval. Prove that the sequence of functions

$$f_n(x)=\frac{f(x+a_n)-f(x)}{a_n}$$

converges uniformly on $[A,B]$ to $f'(x)$.
[**Hint:** use uniform continuity of f' on $[A,B]$ together with the Mean Value Theorem.]

(b) Define $h\colon\mathbb{R}\to\mathbb{R}$ by

$$h(y)=\int_0^1 g(x)f(x+y)\,dx.$$

Prove that h is differentiable and

$$h'(y)=\int_0^1 g(x)f'(x+y)\,dx.$$

[**Hint:** use (a) and the Proposition 6.6.1.]