MATH 142B
SPRING 2020
SECTION B00 (MANNERS)

Homework – week 1

Due by 2359 (11:59 PM) on Tuesday April 7. Hand in via Gradescope.

You may discuss these problems among yourselves, but your final submitted solutions must be written by you alone.

1. For each of the following power series, (i) find the radius of convergence R, and (ii) if applicable, determine whether the series converges and/or converges absolutely at $x = \pm R$.

(a) $f_1(x) = \sum_{n=0}^{\infty} n^2 x^n$;

(b) $f_2(x) = \sum_{n=0}^{\infty} 2^{n^2} x^n$;

(c) $f_3(x) = \sum_{n=0}^{\infty} x^{n^2}$;

(d) $f_4(x) = \sum_{n=0}^{\infty} a_n x^n$ where

$$a_n = \begin{cases} 2^n & : n \text{ is even;} \\ 3^n & : n \text{ is odd.} \end{cases}$$

2. Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence 1. Which of the following statements (taken individually) could be true? In each case given either an example or a disproof.

(a) f converges absolutely at both $x = 1$ and $x = -1$.

(b) f converges absolutely at $x = 1$ but does not converge at $x = -1$.

(c) f converges *conditionally* at both $x = 1$ and $x = -1$.

3. Suppose I have two power series

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$G(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Define coefficients c_n by $c_n = \sum_{m=0}^{n} a_m b_{n-m}$; so

$$c_0 = a_0 b_0$$
$$c_1 = a_0 b_1 + a_1 b_0$$
$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$$
$$\vdots$$

Finally set $H(x)$ to be the power series

$$H(x) = \sum_{n=0}^{\infty} c_n x^n.$$

(a) Show carefully that, for any fixed integer $N \geq 0$ and any real number $x \in \mathbb{R}$:

$$\left| \left(\sum_{n=0}^{N} a_n x^n \right) \left(\sum_{n=0}^{N} b_n x^n \right) - \sum_{n=0}^{N} c_n x^n \right| \leq \left(\sum_{n=0}^{N} |a_n||x|^n \right) \left(\sum_{n=\lfloor N/2 \rfloor}^{N} |b_n||x|^n \right) + \left(\sum_{n=\lfloor N/2 \rfloor}^{N} |a_n||x|^n \right) \left(\sum_{n=0}^{N} |b_n||x|^n \right).$$

(b) Suppose $x \in \mathbb{R}$ and $|x| < R_1$, $|x| < R_2$, where R_1, R_2 are the radii of convergence of F and G respectively. Show that $H(x)$ converges and that $H(x) = F(x)G(x)$.
[**Hint:** use your answer to (a) and send $N \to \infty$.]

(c) Prove that the radius of convergence of H is at least $\min(R_1, R_2)$.
[**Hint:** it is probably not helpful to use the $\limsup |c_n|^{1/n}$ characterization – use your ansnwer to (b) instead.]

(d) Can the radius of convergence of H be bigger than $\min(R_1, R_2)$? Give an example or a disproof.