Notes on a simple Expanding Universe

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1 Introduction

These are notes of calculations to understand signals and horizons in an expanding universe. This involves only the geometrical side of General Relativity applied to the FLRW[1] Metric in dimension two (one time and one space). Note that this is geometry, not physics.

There are [three?] pages of figures: Signals, Horizons, and Problem. These are located prior to the reference at the end of these notes.

2 The Metric

The FLRW Metric appears in several forms. The *conventional form*, the most frequently used, is

$$ds^2 = -\,c^2\,dt^2 + a(t)^2\left(\frac{1}{1-kr^2}\,dr^2 + r^2\,d\varphi^2 + r^2\,\sin^2\varphi\,d\theta^2\right) \quad ,$$

but we immediately turn to the *geometric form* which is easiest to understand:

$$ds^2 = -\,c^2\,dt^2 + a(t)^2\left(d\rho^2 + S(k;\rho)^2\,d\varphi^2 + S(k;\rho)^2\,\sin^2\varphi\,d\theta^2\right) \quad ,$$

by substituting $r = S(k;\rho)$, where $S(k;\rho)$ is $\sin\rho$, ρ, or $\sinh\rho$, according as $k = 1, 0, \text{or} -1$, respectively.

Here

[1] Friedmann, Lamaître, Robertson, Walker

- c : the speed of light, m/s (meters/second)
- t : cosmic time, s (seconds)
- k : "curvature", unitless
- θ, φ : two angels for "spherical coördinates , r (radians)
- ρ : third spherical coördinate, r (radians)
- $a(t)$: scale factor for expansion of distances, m/r (meters/radian) .

In case $k = 1$, the spatial part if the space-time is a three-sphere of radius $a(t)$, and the use of radians is most appropriate.

We will be concerned only with dimension two (one time, one space). The geometric form of the metric is then

$$ds^2 = -\,c^2\,dt^2 + a(t)^2 d\rho^2 \quad , \tag{$d2$}$$

where space is of one Euclidean dimension and has coördinate ρ.

Standing assumption: $a(t)$ is > 0 and is differentiable.

Looking ahead, we will introduce

- ρ : *comoving distance*, r (radians)
- $x = a(t)\rho$: *proper distance*, m (meters)

Cosmic time $t = 0$ is the present, the past and future are t negative and t positive, respectively. The observer is at $\rho = 0$, that is, at $x = 0$. The speed of light $c > 0$ is positive for photons leaving the observer, and negative $c < 0$, same magnitude, for photons approaching the observer. Square roots will always be non-negative.

Units. It is useful to use units to check formulas. Occasionally we follow a formula with its units. For example, the units for the metric ($d2$) will be written

$$m^2 = -\left(\frac{m}{s}\right)^2 s^2 + \left(\frac{m}{r}\right)^2 r^2 \quad . \qquad \text{units } (d2)$$

The usual signs: $+$, $-$, and $=$, are merely for the purpose of locating the terms to which the units refer.

3 Geodesics

3.1 Connection

To obtain curvature and geodesics, one needs the Christoffel symbols of the connection. Those of the *first kind* are obtained from the metric g_{ij} by:

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) \quad ,$$

symmetric in i and j. Christoffel symbols of the *second kind* are:

$$\Gamma^k_{ij} = \sum_\kappa g^{k\kappa}[ij,\kappa] \quad ,$$

where $[g^{ij}] = [g_{ij}]^{-1}$, matrix inverse, also symmetric in i and j,.

These formulas are to be found in any treatise on Riemannian geometry, or on General Relativity.

3.2 Equations for geodesics

A curve α is a *geodesic* if its tangent is parallel along the curve. This is the analog of unaccelerated motion in Euclidean space. The coöordinates $x^k = \alpha^k(\lambda)$ of the geodesic α satisfy the system of second-order equations

$$\frac{d^2x^k}{d\lambda^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = 0 \quad ,$$

for all k. Note that the Christoffel symbols can contain all of the dependant variables: $\Gamma^k_{ij}(..,x^h,...)^{\cdot}$.

3.3 Dimension two

In space-time of dimension two a geodesic α is given by parameters: $t = \alpha^0(\lambda)$ and $\rho = \alpha^1(\lambda)$. The two equations for this geodesic are

$$\frac{d^2t}{d\lambda^2} + \Gamma^0_{00}\left(\frac{dt}{d\lambda}\right)^2 + 2\,\Gamma^0_{01}\,\frac{dt}{d\lambda}\,\frac{d\rho}{d\lambda} + \Gamma^0_{11}\left(\frac{d\rho}{d\lambda}\right)^2 = 0 \qquad (1)$$

and

$$\frac{d^2\rho}{d\lambda^2} + \Gamma^1_{00}\left(\frac{dt}{d\lambda}\right)^2 + 2\,\Gamma^1_{01}\,\frac{dt}{d\lambda}\,\frac{d\rho}{d\lambda} + \Gamma^1_{11}\left(\frac{d\rho}{d\lambda}\right)^2 = 0 \qquad (2)$$

When $dt/d\lambda \neq 0$ one can eliminate the parameter[2] λ and obtain a differential equationfor ρ in terms of t. The result[3] is

$$\frac{d^2\rho}{dt^2} - \Gamma^0_{11}\left(\frac{d\rho}{dt}\right)^3 + (\Gamma^1_{11} - 2\,\Gamma^0_{01})\left(\frac{d\rho}{dt}\right)^2 + (2\,\Gamma^1_{01} - \Gamma^0_{00})\,\frac{d\rho}{dt} + \Gamma^1_{00} = 0 \quad .$$

This is an important formula for the geometry of surfaces.

4 Hubble

4.1 General

The two dimensional metric $(d2)$: $ds^2 = -\,c^2\,t^2 + a(t)^2\,d\rho^2$ has coördinates $x^0 = t$ and $x^1 = \rho$. The only non-zero Christoffel symbols are:

$$\Gamma^0_{11} = \frac{a\dot{a}}{c^2}\,, \quad \Gamma^1_{01} = \frac{\dot{a}}{a}\,, \quad \text{and} \quad \Gamma^1_{10} = \frac{\dot{a}}{a}\,,$$

where $\dot{a} = da/dt$.

The equation for geodesics becomes

$$\frac{d^2\rho}{dt^2} - \frac{a\dot{a}}{c^2}\left(\frac{d\rho}{dt}\right)^3 + 2\,\frac{\dot{a}}{a}\,\frac{d\rho}{dt} = 0 \quad , \qquad \text{(h)}$$

with units

$$\frac{r}{s^2} - (r^{-2}s)\left(\frac{r}{s}\right)^3 + (s^{-1})\,\frac{r}{s} = 0 \quad . \qquad \text{units } (h)$$

Note that there are no restrictions: a geodesic can be time-like, null, or space-like ($t \neq constant$).

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[2]Into (2) substitute $\frac{d\rho}{d\lambda} = \frac{d\rho}{dt}\,\frac{dt}{d\lambda}$ and $\frac{d^2\rho}{d\lambda^2} = \frac{d^2\rho}{dt^2}\left(\frac{dt}{d\lambda}\right)^2 + \frac{d\rho}{dt}\,\frac{d^2t}{d\lambda^2}$, multiply (1) by $\frac{d\rho}{dt}$, subtract from (2), and divide by $(\frac{dt}{d\lambda})^2$.

[3]D.J.Struik, Lectures on Classical Differential Geometry, 2nd. ed., Dover, 1988. Section 4-2.

One easily checks that the integrals

$$\rho(t) = \rho(t_0) + \int_{t_0}^{t} \frac{c}{a(t')}\, dt'$$

(c positive or negative) are solutions. These solutions are null geodesics.[4] A choice of c and a choice of $\rho(t_0)$ at $t = t_0$ are the two initial conditions for this second order ordinary differential equation. Initial conditions for non-null geodesics will be addessed later.

The time derivative of the logarithm of the expansion factor is known as the *Hubble constant* H. It need not be constant:

$$\frac{\dot{a}(t)}{a(t)} = H(t) \quad \text{and} \quad a(t) = a(t_0)\, exp \int_{t_0}^{t} H(t')\, dt' \quad .$$

The units of $H(t)$ are $(m/rs)/(m/r) = s^{-1}$.

The scale factor $a(t)$ relates the comoving dstance ρ and the proper diestance x at the same cosmic time t by $x = a(t)\, \rho(t)$.

Now, $\dot{a}(t) = H(t)\, a(t)$ and for a null geodesic $\dot{\rho}(t) = c/a(t)$. From $x = a\rho$ we have $\dot{x} = \dot{a}\rho + a\dot{\rho} = Ha\rho + a(\frac{c}{a})$, so

$$\dot{x} = H(t)x(t) + c \quad .$$

This is interpreted as

$$\left(\begin{array}{c} speed\ of\ photon \\ along\ null\ geodesic \end{array} \right) = \left(\begin{array}{c} expansion \\ of\ universe \end{array} \right) + \left(\begin{array}{c} speed \\ of\ light \end{array} \right)$$

From the view point of the comoving distance ρ, light is slowing down: $\dot{\rho}(t) = c/a(t)$.

4.2 Quadrature

We return to the equation for a geodesic

$$\frac{d^2\rho}{dt^2} - \frac{a\dot{a}}{c^2}\left(\frac{d\rho}{dt}\right)^3 + 2\,\frac{\dot{a}}{a}\,\frac{d\rho}{dt} = 0 \quad , \tag{h}$$

[4]For $\dot{\rho} = c/a(t)$, and from $(d2)$ we have $(\frac{ds}{dt})^2 = -\,c^2 + a(t)^2\,\dot{\rho}^2$. Hence, $(ds/dt)^2 = 0$.

where $\dot{a} = da/dt$. And, $a(t)$ is known. There are no restrictions: a geodesic can be time-like, null, or space-like ($t \neq constant$).

It is remarkable that this equation is solvable by quadrature – that is, the solution is expressable as an integral, but the integral may or may not be possible to write it in terms of well-known functions.

Set $u(t) = \dot{\rho}(t)/c$. The equation becomes

$$\frac{du}{dt} - a\dot{a}\,u^3 + 2\,\frac{\dot{a}}{a}\,u = 0 \quad .$$

Multiplying by $a(t)^2$ we have

$$\frac{d}{dt}(a^2\,u) - a^3\dot{a}\,u^3 = 0 \quad \text{and} \quad \frac{d}{dt}(a^2\,u) = (a^2\,u)^3\,\frac{1}{a^3}\,\frac{da}{dt} \quad .$$

Write this last equation as

$$\frac{d(a^2u)}{(a^2u)^3} = \frac{da}{a^3} \quad .$$

We integrate formally, and check later that the solution is valid. (It may be that a cannot be an independent variable.) We have:

$$-\frac{1}{2}\,\frac{1}{(a^2u)^2} = -\frac{1}{2}\,\frac{1}{a^2} + constant \quad .$$

Denote the consant of integration by $constant = \frac{1}{2}\,B$. Then

$$B = \frac{1}{a^2} - \frac{1}{a^4\,u^2} \quad . \tag{0}$$

Recalling that $u = \dot{\rho}/c$, (0) becomes

$$B = \frac{1}{a(t)^2}\left(1 - \frac{c^2}{a(t)^2\dot{\rho}(t)^2}\right) \quad , \tag{1}$$

which is a constant; and solving (0) for u gives

$$\dot{\rho}(t) = \frac{c}{a(t)}\,\frac{1}{\sqrt{1 - Ba(t)^2}} \quad . \tag{2}$$

(Keep in mind that c may be positive or negative.)

Equation (1) evaluated at $t = t_0$ shows that $\dot{\rho}(t_0)$ determines the constant B. Equation (2) evaluated at $t = t_0$ shows that B detrmines $\dot{\rho}(t_0)$.

And, one may now check, by substitution, that (2) is indeed a solution to the differential equation (h).

Note that the constant B has units $m^{-2}r^2$, and Ba^2 is unitless.

From $d\rho = \dot{\rho}\, dt$, equation (2), and the metric $(d2)$, we have

$$\left(\frac{ds}{dt}\right)^2 = c^2\, \frac{Ba^2}{1 - Ba^2} \quad .$$

So, the kind of geodesic is determined[5] by B:

geodesic	$(ds/dt)^2$	Ba^2
timelike	$= -c^2$ $-c^2 < (\ )^2 < 0$	$Ba^2 = \pm\infty$ $-\infty < Ba^2 < 0$
null	$= 0$	$Ba^2 = 0$
spacelike	> 0	$0 < Ba^2 < 1$
none		$1 \leq Ba^2 < \infty$

Note:

- The radicands are positive for $-\infty < Ba^2 < 1$.
- For $Ba^2 = \pm\infty$, $\rho(t)$ is constant.

Aside. One also has $1 - Ba^2 = \frac{c^2}{a^2\dot{\rho}(t_0)^2}$ with which one can compare the speed of light $a\,\dot{\rho}(t_0)$, as viewed in the expanding comoving coördinates, with c.

Finally, $\rho(t)$ is detrmined by

$$\rho(t) = \rho(t_0) \; + \; \int_{t_0}^{t} \frac{c}{a(t')}\, \frac{1}{\sqrt{1 - Ba(t')^2}}\, dt' \quad .$$

Earlier we had observed that the integrals

$$\rho(t) = \rho(t_0) \; + \; \int_{t_0}^{t} \frac{c}{a(t')}\, dt'$$

are solutions to (h) – null geodesics. These are the special case $B = 0$.

[5]It is worth sketeching $y = x/(1 - x)$ for this table.

4.3 Exponential

To obtain a non-trivial example in which all calulations may be carried out explicitly, we assume that the Hubble constant is indeed a constant H:

$$a(t) = a(0)\, e^{Ht} \quad .$$

Null geodesics: $Ba^2 = 0$, the paths of photons.

Here we will use $a(t) = a(z)\, e^{H(t-z)}$ with $t = z$ for the present – allowing the possibility of choosing a different time for the present. This will also affirm that the presence of z will not affect the exponential formulas, and that we may later use $z = 0$.

From $\dot{\rho}(t) = c/a(t)$,

$$\dot{\rho}(t) \;=\; \frac{c}{a(z)e^{H(t-z)}} \;=\; \frac{c}{a(z)e^{-Hz}}\,\frac{1}{e^{Ht}} \quad .$$

Then $\rho(t)$ is determined by

$$\begin{aligned}\rho(t) - \rho(t_0) \;&=\; \frac{c}{a(z)e^{-Hz}} \int_{t_0}^{t} \frac{1}{e^{Ht'}}\, dt' = \frac{c}{a(z)e^{-Hz}}\,\frac{1}{H}\left(e^{-Ht_0} \,-\, e^{-Ht}\right) \\ &= \frac{c}{a(z)e^{-Hz}}\,\frac{1}{e^{Ht_0}}\,\frac{1}{H}\left(1 \,-\, e^{-H(t-t_0)}\right) = \frac{c}{a(t_0)}\,\frac{1 \,-\, e^{-H(t-t_0)}}{H} \quad .\end{aligned}$$

And, to determine $x(t)$:

$$x(t) = a(t)\,\rho(t)\;, \quad a(t) = a(t_0)\, e^{H(t-t_0)}\;, \quad \rho(t) = \rho(t_0) \,+\, \frac{c}{a(t_0)}\,\frac{1 \,-\, e^{-H(t-t_0)}}{H}$$

yields

$$x(t) \;=\; x(t_0)\, e^{H(T-t_0)} \;+\; c\,\frac{e^{H(t-t_0)} \,-\, 1}{H} \quad .$$

Aside. For $H(t - t_0)$ small, we can expand and obtain

$$\rho(t) - \rho(t_0) = \frac{c}{a(t_0)}\left((t-t_0) - \frac{1}{2}H(t-t_0)^2 + \frac{1}{6}H^2(t-t_0)^3 - \dots\right) \quad ,$$

and then $\rho(t) - \rho(t_0) \approx \frac{c}{a(t_0)}\,(t - t_0)$.

Timelike geodesics: $-\infty < Ba^2 < 0$, the paths of massive particles. Now we use $a(t) = a(0)\, e^{Ht}$.

From

$$\dot{\rho}(t) = \frac{c}{a(t)} \frac{1}{\sqrt{1 - Ba(t)^2}} \quad .$$

we have

$$\dot{\rho}(t) \;=\; \frac{c}{a(0)e^{Ht}} \frac{1}{\sqrt{1 - B(a(0)e^{Ht})^2}} \;=\; \frac{c}{a(0)} \frac{1}{e^{Ht}} \frac{1}{\sqrt{1 - \beta e^{2Ht}}} \quad ,$$

where

$$\beta = Ba(0)^2 \quad .$$

Then $\rho(t)$ is detrmined by

$$\begin{aligned}
\rho(t) - \rho(t_0) \;&=\; \frac{c}{a(0)} \int_{t_0}^{t} \frac{1}{e^{Ht'}} \frac{1}{\sqrt{1 - \beta e^{2Ht'}}}\, dt' \\
&= \frac{c}{a(0)} \frac{1}{H} \left[-\; \frac{1}{e^x} \sqrt{1 - \beta\, e^{2x}} \right]_{x=Ht_0}^{x=Ht} \\
&= \frac{c}{a(0)} \sqrt{1 - \beta\, e^{2Ht_0}} \frac{1}{e^{Ht_0}} \frac{1}{H} \left(1 \;-\; e^{-H(t-t_0)} \frac{\sqrt{1 - \beta\, e^{2Ht}}}{\sqrt{1 - \beta\, e^{2Ht_0}}} \right) \\
&= \frac{c}{a(t_0)} \sqrt{1 - \beta\, e^{2Ht_0}} \; \frac{1 \;-\; e^{-H(t-t_0)} \frac{\sqrt{1-\beta\, e^{2Ht}}}{\sqrt{1-\beta\, e^{2Ht_0}}}}{H} \quad .
\end{aligned}$$

And, $x(t)$ can be determined in a manner like the above.

Timelike geodesics: $Ba^2 = -\infty$, the paths of stationary particles.

Here $\dot{\rho}(t) = 0$, so $\rho(t) = \textit{constant}$, and $x(t)$ is increasing ($H > 0$).

5 Signals

5.1 Example - exponentially expanding universe

For an example of an expanding universe we will use $a(t) = a(0)\, e^{Ht}$, $H > 0$ a constant.

The space-time graphs are drawn with the independent variable t (cosmic time) as the ordinate (verticle), and the dependent variable ρ or x as the abscissa (horizontal).

Comoving distance ρ. See Figure 1. A signal is sent along a null-geodesic from location $\rho = 0$ at time t_0 and arrives at location $\rho = \rho_1$ at time t_1. The time t_1 is to be determined.

We have

$$\dot{\rho}(t) = \frac{c}{a(t_0)\, e^{H(t-t_0)}} \quad \text{and} \quad \rho(t) = \frac{c}{a(t_0)}\, \frac{1 - e^{-H(t-t_0)}}{H} \quad .$$

The least distance to which a signal may not arrive is $\rho = c/a(t_0)H$.

The arrival time is

$$t_1 = t_0 + \frac{1}{H}\, \ln\, \frac{1}{1 - \frac{a(t_0)\, \rho_1}{c}\, H} \quad \text{for} \quad a(t_0)\, \rho_1 < \frac{c}{H} \quad .$$

For $\rho_1 \geq c/a(t_0)H$, there is no solution.

To pass from comoving distances to proper distances we use [box]$x = a(t)\, \rho$. In the case of this expanding universe $a(t) = a(t_0)\, e^{H(t-t_0)}$ meters/radian.

Proper distance x. See Figure 2. A signal is sent along a null-geodesic from location $x = 0$ at time t_0 and arrives at location $x = x_1$ at time t_1. The time t_1 is to be determined.

Now

$$x(t) = a(t_0)\, e^{H(t-t_0)} \times \frac{c}{a(t_0)}\, \frac{1 - e^{-H(t-t_0)}}{H} = c\, \frac{e^{H(t-t_0)} - 1}{H} \quad .$$

Then t_1 is determined by solving

$$c\, \frac{e^{H(t_1-t_0)} - 1}{H} \;=\; x_1\, e^{H(t_1-t_0)} \quad .$$

The result is

$$t_1 = t_0 + \frac{1}{H}\, \ln\, \frac{1}{1 - \frac{x_1}{c}\, H} \quad \text{for} \quad x_1 < \frac{c}{H} \quad .$$

For $x_1 \geq c/H$ there are no solutions.

Observe that the formulas for ρ and x are entirely consistant.

If one examines these formulas as $H \to 0$, formulas for the case a static universe will be recognised, [6] especially $x(t) = c(t - t_0)$.

5.2 General - expanding universe

For the general case of an expandong (and contracting) universe, we assume only that $a(t)$ is > 0.

The space-time graphs will be visualized with the independent variable t (cosmic time) as the ordinate (verticle), and the dependent variable ρ or x as the abscissa (horizontal).

Comoving distance ρ. A signal is sent along a null-geodesic from location $\rho = 0$ at time t_0 and arrives at location $\rho = \rho_1$ at time t_1. The time t_1 is to be determined.

For null geodesics $\dot{\rho}(t) = c/a(t)$, so $d\rho = c\,\frac{dt}{a(t)}$. Then

$$\rho(t) = c \int_{t_0}^{t} \frac{dt'}{a(t')} \quad .$$

To determine t_1, one solves

$$c \int_{t_0}^{t_1} \frac{dt'}{a(t')} = \rho_1$$

for t_1. If $\rho_1 \geq c \int_{t_0}^{\infty} \frac{dt'}{a(t')}$, as in the example, there is no solution.

To pass from comoving distances to proper distances we [again] use $x = a(t)\,\rho$.

Proper distance x. A signal is sent along a null geodesic from location $x = 0$ at time t_0 and arrives at location $x = a(t_1)\,x_1$ at time t_1. The time t_1 is to be determined.

Now the signal follows the path

$$x = c\,a(t) \int_{t_0}^{t} \frac{dt'}{a(t')}$$

[6] Useful:

$$\frac{1}{H} \ln \frac{1}{1 - uH} \;=\; u + \frac{1}{2}\,u^2\,H + \frac{1}{3}\,u^3\,H^2 + \frac{1}{4}\,u^4\,H^3 + \cdots \quad .$$

and the receiver is at $x = a(t)\,\rho_1$. Equating these at $t = t_1$:

$$c\,a(t_1)\int_{t_0}^{t_1}\frac{dt'}{a(t')} = a(t_1)\,\rho_1 \quad .$$

Cancel $a(t_1)$ to obtain the same formula as before to be solved for t_1 :

$$c\int_{t_0}^{t_1}\frac{dt'}{a(t')} = \rho_1 \quad .$$

5.3 Horizons

It is common to express horizons in terms of comoving coördinates; we do that here.

An event within the *particle horizon* has been oberved in the past or is observed at the present. An event beyond the *event horizon* cannot be observed at any time - including the future. See Figure 3 at the end of this article.

Again the space-time graph are drawn with the independent variable t (cosmic time) as the ordinate (verticle), and the dependent variable ρ as the abscissa (horizontal).

On the ordinate:

- Big Bang: $t = t_0$ which is negative.
- The present: $t = 0$.

A photon travels along a null geodesic from an event at (t_0, ρ_0) and is received at $(t_1, 0)$.

On the abscissa:

- Observer: $\rho = 0$.
- Particle horizon: ρ_{PH}
- Event horizon: ρ_{EH}

We take the speed of light to be negative and write $\dot{\rho}(t) = -\,c/a(t)$ with $c > 0$. Time t_1 and comoving distance ρ_0 are related by

$$0 - \rho_0 = \int_{t_0}^{t_1} -\,\frac{c}{a(t')}\,dt' \quad .$$

One thinks in tems of solving for t_1 given ρ_0.
Especially note the particle horizon: $t_1 = 0$:

$$\rho_{PH} = \int_{-|t_0|}^{0} \frac{c}{a(t')}\, dt' \quad ,$$

and the event horizon $t_1 = \infty$:

$$\rho_{EH} = \int_{-|t_0|}^{\infty} \frac{c}{a(t')}\, dt' \quad ,$$

where $-|t_0|$ emphasizes that t_0 is negative.

ρ	t_1	seen
$0 < \rho < \rho_{PH}$	$t_0 < t_1 < 0$	past
$\rho = \rho_{PH}$	$t_1 = 0$	present
$\rho_{PH} < \rho < \rho_{EH}$	$0 < t_1$	future
$\rho_{EH} \leq \rho$	no solution	never

Example: The familiar $a(t) = a(0)\, e^{Ht}$, $H > 0$.

We have

$$\rho_{PH} \;=\; \int_{t_0}^{0} \frac{c}{a(0)\, e^{Ht'}}\, dt' \;=\; \frac{c}{a(t_0)}\, \frac{1}{H}\, (1 - e^{Ht_0}) \quad ,$$

and

$$\rho_{EH} \;=\; \int_{t_0}^{\infty} \frac{c}{a(0)\, e^{Ht'}}\, dt' \;=\; \frac{c}{a(t_0)}\, \frac{1}{H} \quad ,$$

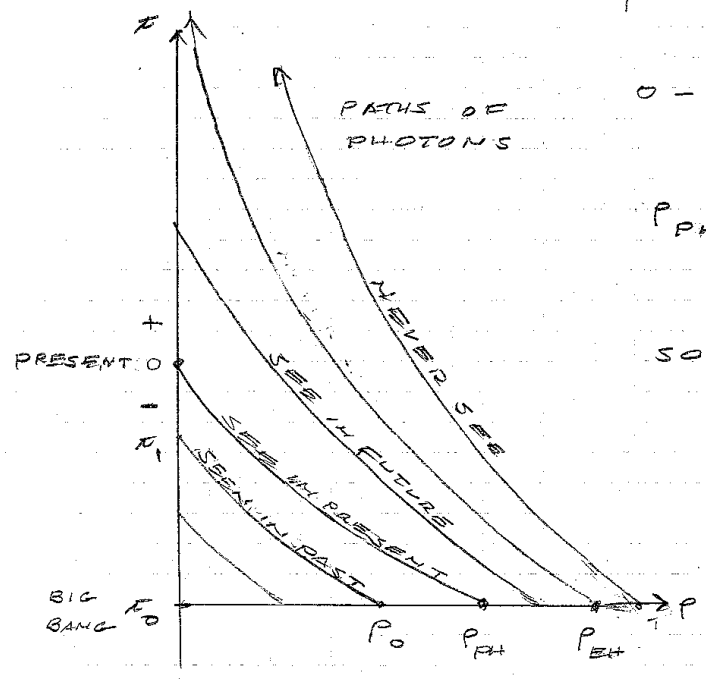
where $Ht_0 < 0$.

We have seen latter in the previous sections: $x = a(t_0)\, \rho_{EH} \;=\; c/H$ determins the distances ρ and x past which solutions for t_1 do not exist. It is commonly called the"cosmological horizon distance".

Problem for future thought. See Figures 3, 4, and 5.

The ship is at O, and a distant galaxy at G. The ship signals from O to G that it is coming for a visit. The signal reaches G and the ship, fairly fast, also reaches G. The ship signals back from G to O that it is returning. The universe is expanding. The signal reaches O, but the ship cannot. A space-time diagram is easy to draw that will convince you that this can happen. The formulas regarding timelike geodisics $-\infty < Ba^2 < 0$ are available. The problem is to determine parameters for this and carry out the necessary calculations.

PARTICLE HORIZON



$$\dot{\rho}(\tau) = -\frac{c}{a(\tau)} \quad , \quad c > 0$$

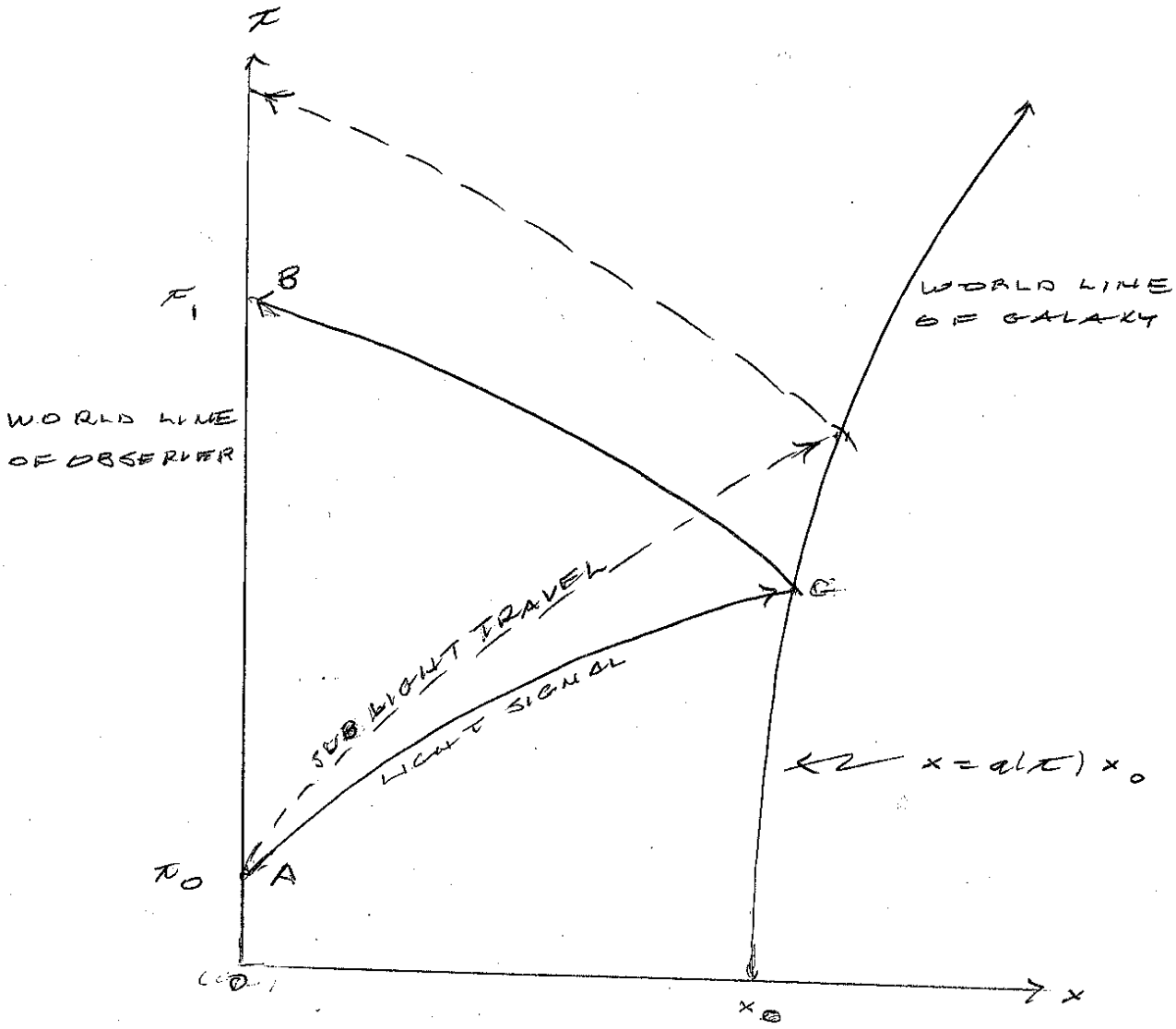
$$0 - \rho_0 = \int_{\tau_0}^{\tau_1} -\frac{c}{a(\tau')}\, d\tau' \quad , \quad \tau_0 < 0$$

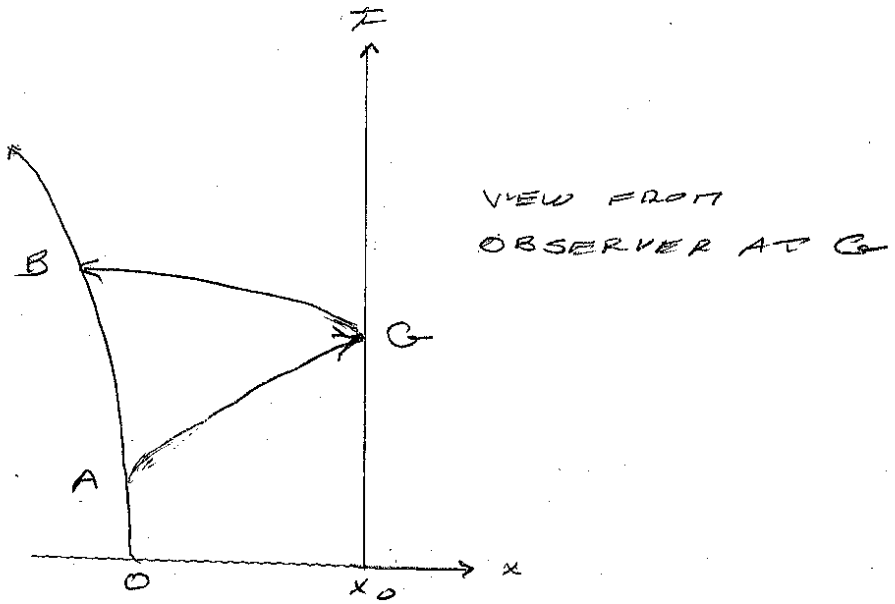
$$\rho_{PH} = \int_{-|\tau_0|}^{0} \frac{c}{a(\tau')}\, d\tau' \qquad \text{PARTICLE HORIZON}$$

SOLVE $\displaystyle\int_{-|\tau_0|}^{\tau_1} \frac{c}{a(\tau')}\, d\tau' = \rho_0$ FOR τ_1:

ρ	τ_1	SEEN
$0 < \rho < \rho_{PH}$	$\tau_0 < \tau_1 < 0$	PAST
$\rho = \rho_{PH}$	$\tau_1 = 0$	PRESENT
$\rho_{PH} < \rho < \rho_{EH}$	$0 < \tau_1$	FUTURE
$\rho_{EH} \le \rho$	NO SOLUTION	NEVER

$$\rho_{EH} = \int_{-|\tau_0|}^{\infty} \frac{c}{a(\tau')}\, d\tau' \qquad \text{EVENT HORIZON}$$

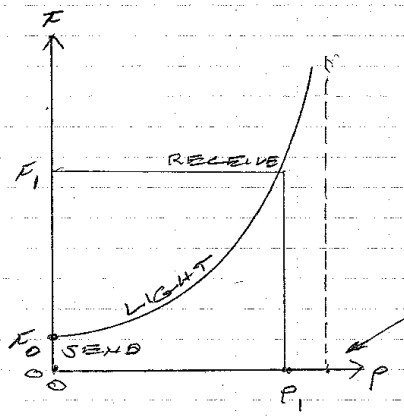




VIEW FROM OBSERVER AT C

SIGNALS - EXPANDING UNIVERSES $a(\tau) = a(\tau_0)\, e^{H\tau}$, $H > 0$ CONSTANT

COMOVING DISTANCE



LIGHT

$$\dot\rho(\tau) = \frac{c}{a(\tau_0)\, e^{H(\tau-\tau_0)}}$$

$$\rho(\tau) = \frac{c}{a(\tau_0)}\,\frac{1-e^{-H(\tau-\tau_0)}}{H}$$

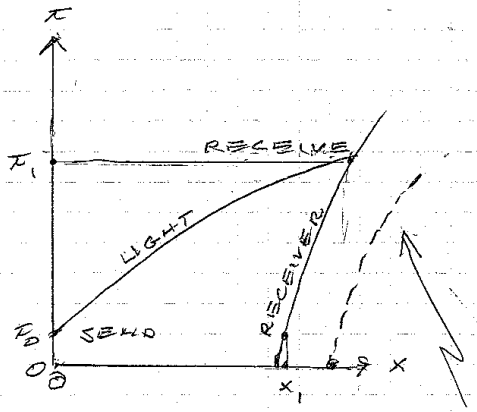
$$\rho(\tau) \to \frac{c}{a(\tau_0)H}$$

$$\tau_1 = \tau_0 + \frac{1}{H}\ln\frac{1}{1-\frac{a(\tau_0)\rho_1 H}{c}}, \qquad \text{FOR } \rho_1 < \frac{c}{a(\tau_0)H}$$

FOR $\rho \geq \frac{c}{a(\tau_0)H}$, NO SOLUTION

$$\boxed{x = a(\tau)\,\rho}$$

PROPER DISTANCE



LIGHT

$$x(\tau) = c\,\frac{e^{H(\tau-\tau_0)}-1}{H}$$

RECEIVER

$$x(\tau) = x_1\, e^{H(\tau-\tau_0)}$$

$$\therefore\ c\,\frac{e^{H(\tau_1-\tau_0)}-1}{H} = x_1\, e^{H(\tau_1-\tau_0)}$$

FOR $x_1 < \frac{c}{H}$

$$\tau_1 = \tau_0 + \frac{1}{H}\ln\frac{1}{1-\frac{x_1}{c}H}$$

FOR $x_1 \geq \frac{c}{H}$

NO SOLUTION

$$x = \frac{c}{H}\, e^{H(\tau-\tau_0)} \qquad \frac{c}{H}$$

COMOVING DISTANCE

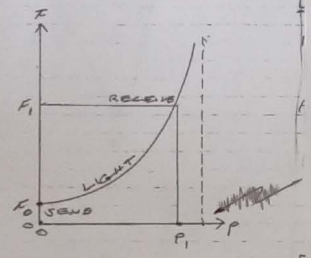


FIGURE 1

PROPER DISTANCE

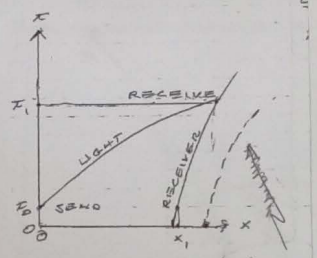


FIGURE 2

PARTICLE HORIZON

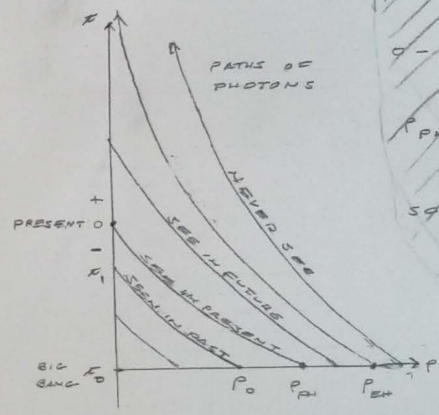


FIGURE 3

$\dot{p}(x) = -$

$\dot{q} - p_0 =$

$p_{PH} =$

SOLVE

p

$0 < p$

$p =$

p_{PH}

p_{BI}

$p_{BH} =$

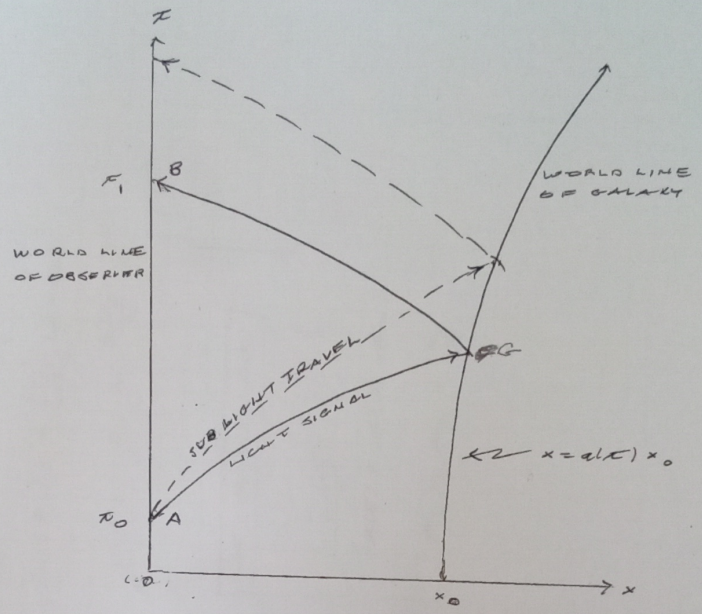


FIGURE 4

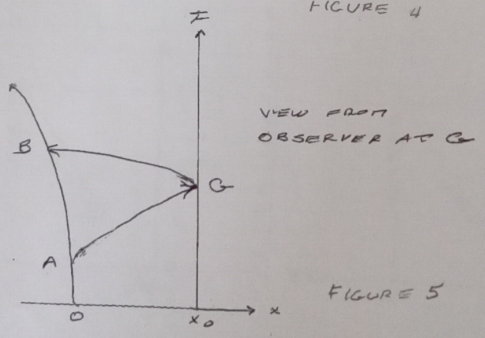


FIGURE 5

Reference

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