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NEW EUCLIDEAN THEOREMS BY THE USE OF LAGUERRE TRANSFORMATIONS - SOME GEOMETRY OF MINKOWSKI (2+1)-SPACE

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Examples of the use of Laguerre transformations to discover theorems in the Euclidean and Minkowski planes.

1. INTRODUCTION

1.1. In a Euclidean plane, an oriented line will be called a spear, an oriented circle will be called a proper cycle, and a point will be called a point cycle. The term cycle will include proper cycles and point cycles. Every line underlies two spears, every circle underlies two proper cycles. Two proper cycles or a proper cycle and a spear touch if the underlying circles or circle and line are tangent and the orientations agree at the point of tangency. Two spears touch if the underlying lines are parallel and the orientations agree. A point cycle touches a proper cycle or a spear if, as a point, it is incident with the underlying circle or line.

Classical Laguerre plane geometry uses spears as primitives, and cycles as envelopes of spears. A Laguerre transformation sends spears to spears, cycles to cycles, and preserves the relation "touch". Such transformations preserve (up to a positive scale factor) a "separation" which extends the notion of Steiner power (of a point and a circle) to two cycles.

Laguerre [6] clearly defined these transformations in 1882, and Pedoe [7 and 8] referred to them as "forgotten geometric transformations" in the 1970s. We are not aware that there appears in the literature any systematic application of these transformations either to obtain theorems or to simplify proofs in Euclidean geometry.

1.2. A key feature of Laguerre geometry, for our purposes, is that it can be represented by the metric affine geometry of Minkowski space (two space dimensions, one time dimension). Points of Minkowski space correspond to cycles, and separation of cycles is the square of spatial distance or proper time (with appropriate signs). Configurations in a Euclidean plane are interpreted as configurations in Minkowski space. Laguerre transformations of the Euclidean plane become exactly the isometries or similarities of Minkowski space .

In this paper we show how to make use of Laguerre transformations, both from the classical viewpoint and from their representation in Minkowski space, to obtain new and striking theorems in Euclidean geometry. We focus on one theorem whose proof exhibits these ideas: The Laguerre transformation of the Pythagorean Theorem. From this one sees how to enunciate other theorems easily in both the Euclidean plane and the Minkowski plane (one space dimension, one time dimension). This we do in the concluding sections.

The authors are indebted to the referee for several suggestions leading to simplification and clarification.

2. SECANT SQUARE-SEPARATION

2.1. The two orientations of a proper cycle are described by attaching a sign to the radius of the underlying circle. (We assign positive radius to proper cycles in the Euclidean plane which are oriented counter-clockwise.) Two proper cycles have the "same" or "opposite" orientations according as these radii have the same or opposite signs.

2.2. Two proper cycles of different radii have exactly one center of similitude. This point cycle touches (at most) two spears which touch the proper cycles. When the radii are equal, the point at infinity on the line of centers is the center of similitude.

Let a line through the center of similitude Z of the two proper cycles A and A' meet the circle underlying A in points P and Q, and meet the circle underlying A' in P' and Q'. Assume these labeled so that PQ and $P'Q'$ are homothetic from Z. Then the product $PP'.QQ'$ of signed distances does not depend on the position of the line through Z. See Court [4, p.186]. We will call this quantity the secant square-separation (or just separation) of cycles A and A'. See Figure 1.

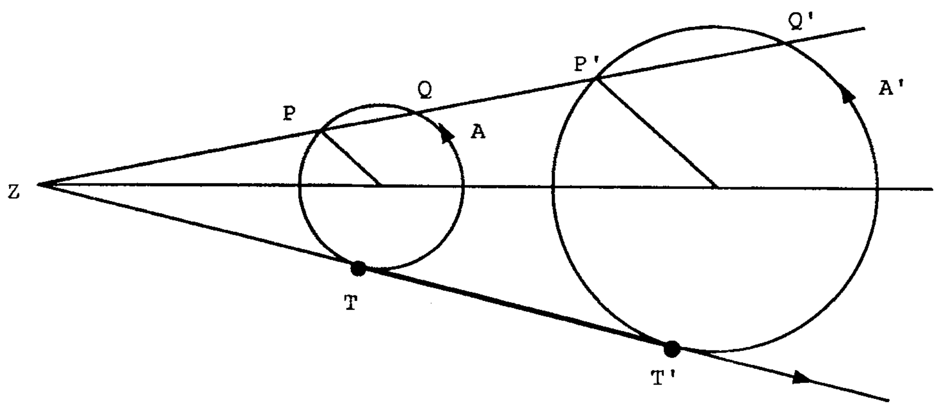

Figure 1. Secant square-separation and tangential distance

2.3. In case the point Z lies outside the circles, the length of the segment TT' between the points of tangency T and T' with a spear through Z is the tangential distance between the two cycles. In this case, the separation is the square of the tangential distance.

2.4. The separation of a point cycle and a proper cycle is defined to be the Steiner power of the point with respect to the underlying circle.

2.5. In general, the separation of two cycles is the square of the distance between their centers less the square of the difference of their (signed) radii.

Two cycles touch exactly when their separation is zero.

2.6. Under a general Laguerre transformation, the separation of cycles is multiplied by a positive factor depending only on the transformation. This will be seen in 5.8. Laguerre transformations which preserve separation are known as "restricted" or "equilong".

3. THE THEOREM OF PYTHAGORAS-LAGUERRE - CLASSICAL DISCOVERY

3.1. On a circle in the Euclidean plane, chose two diametrically opposite points A and B, and a third point C. The circle underlies two cycles R and S of opposite orientation. These cycles touch the point cycles A, B, C. The line tangent to the circle at A underlies two spears U and U' which touch R and S, respectively. Similarly, from B obtain two spears V and V' which touch R and S, respectively. Since AB is a diameter of the circle, spears U and V' touch, and spears U' and V touch.

Suppose a Laguerre transformation sends R and S to two proper cycles. Five cycles and four spears are obtained. We use again the letters A, B, C, R, S to denote the image cycles, and U, U', V, V' to denote the image spears. Let X and X' be the point cycles where A touches R and S, respectively. Then, X and X' touch U and U', respectively. Similarly, let Y and Y' be the point cycles obtained from B. See Figure 2.

The lines underlying U and U' make equal angles with line XX' , as do the lines underlying V and V' . Now, two tangents to a circle make equal angles with the line joining the points of tangency. Since U and V' touch, and U' and V touch, the points X,X',Y,Y' are collinear.

Since a Laguerre transformation multiplies separation by a positive factor, we are led to:

3.2. THEOREM (Pythagoras - Laguerre) Let A, R, B, S be four cycles such that each touches the next (cyclically) with the four points of tangency being collinear, and R and S having opposite orientations. If C is any cycle touching R and S , then the square of the tangential distance from A to B is the sum of the squares of the tangential distances from B to C and from C to A . See Figure 2.

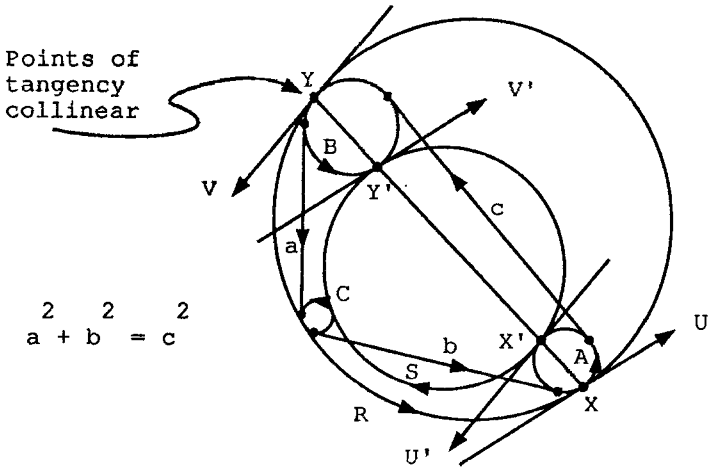

Figure 2. Laguerre transformation of the Pythagorean Theorem

There is a Euclidean proof of this particular theorem, but it is not completely trivial - even in the special case that cycles R and S have the same radius, opposite orientations, and meet in point cycles A and B .

This classical use of Laguerre transformations was just to lead us to such a theorem. Its proof follows from the fact that every such configuration of five cycles can be obtained from a Laguerre transformation of the configuration of the Pythagorean theorem. This will be immediately evident from the viewpoint of Minkowski geometry.

4. MINKOWSKI GEOMETRY

4.1. Let M^3 be three-dimensional Minkowski space. We regard M^3 as a metric affine space, as it is important not to give preferential treatment to any one Euclidean plane in M^3. The vector from the point P to the point Q is denoted $\overrightarrow{PQ}$. The inner product on the space of vectors is denoted $(\vec{x}|\vec{y})$ and has signature $(++-)$.

This is the familiar geometry of special relativity in which a line PQ is time-like, light-like, or space like according as $(\overrightarrow{PQ}|\overrightarrow{PQ})$ is negative, zero, or positive, respectively. Here we merely establish terminology, point out needed facts, and prove one theorem.

4.2. A plane is orthogonal to a time-like line exactly when, as a metric affine plane, it is Euclidean - the inner product restricted to vectors between points of the plane has signature $(++)$. A plane is orthogonal to a space-like line exactly when it is Minkowskian - the inner product restricted to vectors between points of the plane has signature $(+-)$. A plane is orthogonal to a light-like line exactly when it is singular - the inner product restricted to vectors between points of the plane has signature $(+0)$; in this case, the line lies in the plane or is parallel to it.

4.3. Through two distinct points P and Q there pass exactly two, one, or no singular planes according as the line PQ is space-, light-, or time-like.

4.4. All light-like lines passing through a given point A constitute a cone with vertex at A called the light-cone at A .

The intersection of a light-cone with a Euclidean plane is a circle (in the traditional sense) in that plane. This will include the possibility of a point. The intersection of a light-cone with a Minkowski plane is a "circle" (usually referred to by its Euclidean description as an equilateral hyperbola) in that plane. This will include the possibility of two light-like lines which meet.

4.5. The intersection of two light-cones whose vertices lie on a time-like line lies in the Euclidean plane orthogonal to the line joining the vertices and passing through its midpoint. (This follows by reasoning like 4.6 to follow.)

4.6. LEMMA [a] (A.A. Robb, 1936) Let a line which is either space-like or time-like meet the light-cone at A in points P and Q , not necessarily distinct. If Z is any point on this line, then $(\overrightarrow{ZP}|\overrightarrow{ZQ}) = (\overrightarrow{ZA}|\overrightarrow{ZA})$. See Figure 3.

PROOF 1) Let $\vec{s} = \frac{1}{2}\overrightarrow{PQ}$, and let M be the midpoint of the segment PQ . Since P and Q lie on the cone, $(\overrightarrow{AM} - \vec{s}|\overrightarrow{AM} - \vec{s}) = 0$ and $(\overrightarrow{AM} + \vec{s}|\overrightarrow{AM} + \vec{s}) = 0$. The difference of these equations is $4(\overrightarrow{AM}|\vec{s}) = 0$. Thus, the vectors $\overrightarrow{AM}$ and $\vec{s}$ are orthogonal, and $(\overrightarrow{AM}|\overrightarrow{AM}) + (\vec{s}|\vec{s}) = 0$.

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[a] This lemma is attributed to A. A. Robb (1936), who set up a coördinate-free system of axioms for Minkowski geometry starting with a partial order describing "after". (C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation, W. H. Freeman, San Francisco, 1973, p.20.) The usual physical interpretation of 4.6 is: The line ZPQ is the world line of an observer fixed at the spatial origin Z of an inertial frame. A light ray is sent from P to A where it is reflected and arrives back to the observer at Q . The square of the Minkowski distance between the events Z and A is the product of the times from Z to P and from Z to Q . The time intervals depend on the inertial frame, but their product does not.

2) $(\overrightarrow{ZP}|\overrightarrow{ZQ}) = (\overrightarrow{ZM} - \vec{s}|\overrightarrow{ZM} + \vec{s}) = (\overrightarrow{ZM}|\overrightarrow{ZM}) - (\vec{s}|\vec{s}) = (\overrightarrow{ZM}|\overrightarrow{ZM}) + (\overrightarrow{AM}|\overrightarrow{AM})$
$= (\overrightarrow{ZA}|\overrightarrow{ZA})$. QED

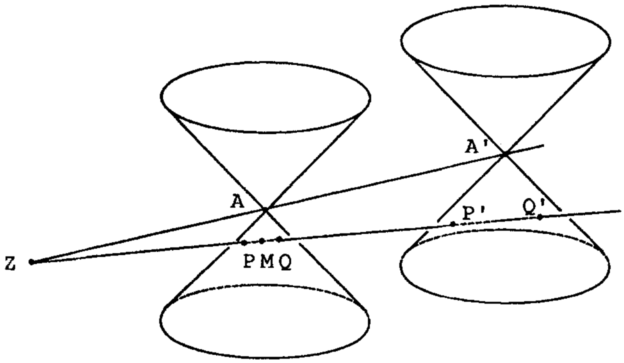

Figure 3. $(\overrightarrow{PP'}|\overrightarrow{QQ'}) = (\overrightarrow{AA'}|\overrightarrow{AA'})$

4.7. The <u>homothety</u> with center Z and ratio $\lambda \neq 0$ is the transformation of M^3 which sends X to X' determined by $\overrightarrow{ZX'} = \overrightarrow{ZX}\,\lambda$. Such a transformation sends lines and planes to lines and planes of the same likeness, and light-cones to light-cones.

4.8. If Z,A,A' are points on a line of M^3 , with Z different from A and A' , then there is a unique homothety with center Z sending A to A' .

4.9. THEOREM Suppose a line ZAA' as in 4.8 is either space-like or time-like. Let any line through Z meet the light-cone at A in points P and Q (not necessarily distinct). This line then meets the light-cone at A' in the image points P' and Q' under the homothety of 4.8. Then: $(\overrightarrow{PP'}|\overrightarrow{QQ'}) = (\overrightarrow{AA'}|\overrightarrow{AA'})$. See Figure 3.

PROOF Note that $\overrightarrow{XX'} = \overrightarrow{XZ} + \overrightarrow{ZX'} = \overrightarrow{ZX}\,(\lambda - 1)$ for any point X and its image X' . Then $(\overrightarrow{PP'}|\overrightarrow{QQ'}) = (\overrightarrow{ZP}|\overrightarrow{ZQ})(\lambda - 1)^2 =$ (by 4.6) $(\overrightarrow{ZA}|\overrightarrow{ZA})(\lambda - 1)^2 = (\overrightarrow{AA'}|\overrightarrow{AA'})$. QED

5. THE REPRESENTATION OF LAGUERRE GEOMETRY

5.1. Laguerre geometry was formulated axiomatically by van der Waerden and Smid in 1935 [10]. A model satisfying such a system of axioms is obtained by representing spears by singular planes and cycles by light cones in Minkowski space. This is the viewpoint of Schaeffer [9] who shows that Laguerre transformations are determined by their effect on cycles. We will work in terms of this model. Intersections of spears and cycles of this model with Euclidean planes leads to classical Laguerre transformations of cycles in the Euclidean plane; intersections with Minkowski planes will lead to Laguerre transformations of "cycles" of the Minkowski plane.

5.2. When referring to the geometry of Minkowski space, we will use the terms "point" and "singular plane"; when referring to Laguerre geometry of a Euclidean plane, we will use the corresponding terms "cycle" (point or proper) and "spear". Except in figures, we will hereafter use the same symbol and underscore the latter.

5.3. Choose and fix any Euclidean plane Π^2_E in a Minkowski space M^3. We will view this plane as the one containing a Euclidean configuration of interest. (Figure 2 is the plane Π^2_E of Figure 4.) Let $\hat{u}$ be a vector orthogonal to Π^2_E for which $(\hat{u}|\hat{u}) = -1$.

5.4. A point A in M^3 represents a cycle $\underline{A}$ in Π^2_E: The underlying circle is the intersection of the light-cone at A with Π^2_E. The foot of the perpendicular from A to Π^2_E is the center A_0 of the circle. The (signed) radius is $-(\hat{u}|\overrightarrow{A_0A})$.

5.5. The center of similitude $\underline{Z}$ of two cycles $\underline{A}$ and $\underline{B}$ is represented by the point Z in which the line AB meets Π^2_E. If this line does not meet the plane, the center of similitude is "at infinity" and the cycles have equal radii. From 4.9, the secant square-separation of cycles $\underline{A}$ and $\underline{B}$ is $(\overrightarrow{AB}|\overrightarrow{AB})$.

5.6. A singular plane in M^3 represents a spear in Π_E^2 : The underlying line is the intersection of the singular plane and Π_E^2 . Any point of the singular plane not on this line represents a proper cycle touching the spear, and thus determines the orientation of the spear. Two spears touch when the planes are parallel.

Two points of M^3 on a common light-like line represent two distinct cycles which touch. The singular plane through this line represents the unique spear that also touches these cycles.

5.7. This representation of Laguerre geometry is classically called the method of "isotropic projection". It is customary to begin with a Euclidean plane E^2 and take the Minkowski space to be $M^3 = E^2 \times R^1$ with inner product $x_1y_1 + x_2y_2 - x_ry_r$. A cycle in $\Pi_E^2 = E^2 \times \{0\}$ has center with coördinates a_1 and a_2 and signed radius a_r . See [1,p.136],[3,p.48],[5,p.248].

5.8. Similarity transformations of M^3 send lines to lines of the same "likeness", planes to planes of the same "likeness", and light-cones to light-cones. These transformations represent Laguerre transformations of the fixed Euclidean plane Π_E^2 : The elements of the Laguerre geometry in Π_E^2 (cycles, spears, secant square-separation,...) are interpreted as elements of Minkowski space M^3 (points, singular planes, the Minkowski metric,...); after a similarity transformation of M^3 , the elements of M^3 are interpreted back in the same Euclidean plane Π_E^2 .

5.9. The foregoing is the "active" viewpoint. We will also use the "passive" viewpoint: The elements of the Laguerre geometry in Π_E^2 are interpreted as elements of Minkowski space M^3 , these elements of M^3 are interpreted in the Laguerre geometry of another Euclidean plane Π^2 . This plane is, in fact, the image of Π_E^2 under the inverse of the similarity transformation of M^3 .

5.10. The transformation known classically as a "Laguerre axial transformation" is a reflection in a non-singular plane of Minkowski space M^3 [1,p.155],[3,p.56],[5,p.254],[7],[8]. Such generate the group of restricted Laguerre transformations.

6. THE THEOREM OF PYTHAGORAS-LAGUERRE - A PROOF

As in 5.3, let Figure 2 be the plane Π^2_E of Figure 4. Our first task is to find a configuration in Minkowski space M^3 which represents the notion of "right triangle" in one Euclidean plane, and which can then be examined in another Euclidean plane.

6.1. Let A and R in M^3 be two points such that the line AR is light-like. Denote by $\Sigma(AR)$ the singular plane containing this line. Then, in the Euclidean plane Π^2_E, the cycles $\underline{A}$ and $\underline{R}$ touch each other, touch the unique point cycle $\underline{X}$ represented by $X = \Pi^2_E \cap AR$, and touch the unique spear $\underline{U}$ represented by $U = \Pi^2_E \cap \Sigma(AR)$.

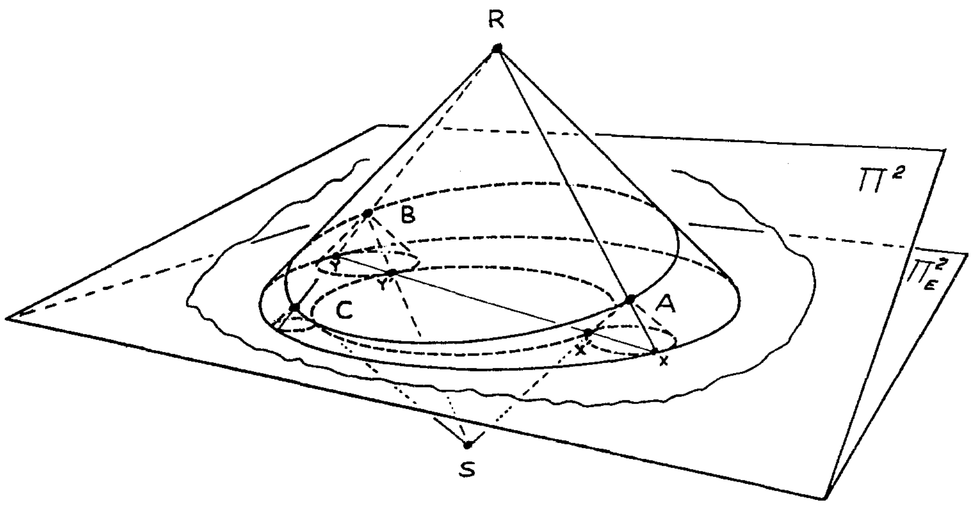

Figure 4. Perspective view of Pythagoras-Laguerre

6.2. Consider now four points A, R, B, S in M^3 such that the lines AR, BR, BS, AS are light-like, and the line RS is time-like. Introduce

$X = \Pi_E^2 \cap AR$ $\quad X' = \Pi_E^2 \cap AS$ $\quad Y = \Pi_E^2 \cap BR$ $\quad Y' = \Pi_E^2 \cap BS$,

$U = \Pi_E^2 \cap \Sigma(AR)$ $\quad U' = \Pi_E^2 \cap \Sigma(AS)$ $\quad V = \Pi_E^2 \cap \Sigma(BR)$ $\quad V' = \Pi_E^2 \cap \Sigma(BS)$.

With these assumptions, the following five assertions are equivalent:

Regarding Minkowski space M^3

- Points A,R,B,S are coplanar (but not collinear).
- Lines AR and BS are parallel
  (or lines BR and AS are parallel).
- Planes $\Sigma(AR)$ and $\Sigma(BS)$ are parallel
  (or planes $\Sigma(BR)$ and $\Sigma(AS)$ are parallel).

Regarding spears and cycles of the Euclidean plane Π_E^2

- Spears $\underline{U}$ and $\underline{V}'$ touch
  (or spears $\underline{U}'$ and $\underline{V}$ touch).
- The point cycles $\underline{X}$, $\underline{X}'$, $\underline{Y}$, $\underline{Y}'$ touch a common spear.

6.3. If Π^2 is any Euclidean plane in Minkowski space M^3 , the assertions of 6.2 hold with Π_E^2 replaced by Π^2 .

6.4. In particular, the plane Π^2 in which the light cones at R and S intersect is Euclidean since line RS is time-like. The circles underlying cycles $\underline{R}$ and $\underline{S}$ in Π^2 then coincide. The last two assertions of 6.2, when interpreted in Π^2 , become:

Regarding the Euclidean plane Π^2

- One line underlies both spears $\underline{U}$ and $\underline{U}'$, and it is parallel to the one line underlying spears $\underline{V}$ and $\underline{V}'$.
- The point cycles $\underline{X}$ and $\underline{X}'$ coincide, as do $\underline{Y}$ and $\underline{Y}'$.

The points underlying these two point cycles lie on a diameter of the circle underlying $\underline{R}$ and $\underline{S}$.

6.5. Proof of 3.2. Let $\underline{A}$, $\underline{R}$, $\underline{B}$, $\underline{S}$ be four cycles of Π^2_E such that each touches the next (cyclically), the four points of tangency are collinear, and cycles $\underline{R}$ and $\underline{S}$ have negative secant square-separation; and let $\underline{C}$ be a cycle touching $\underline{R}$ and $\underline{S}$. As in 6.4, let Π^2 be the Euclidean plane in which the light-cones at R and S intersect. This plane contains the points A, B, C as point cycles. From the Pythagorean theorem in Π^2, we have $(\overrightarrow{AB}|\overrightarrow{AB}) = (\overrightarrow{AC}|\overrightarrow{AC}) + (\overrightarrow{CB}|\overrightarrow{CB})$. By 5.5, this is the sum of separations of cycles in the plane Π^2_E. QED.

6.6. Figure 4 [b] shows the light-cones at A, B, C, R, S and the cycles they represent on planes Π^2_E and Π^2. This is the passive interpretation of a Laguerre transformation which takes the configuration of the theorem to that of the classical theorem of Pythagoras [c].

7. ADDITIONAL EXAMPLES IN THE EUCLIDEAN PLANE

7.1. In M^3, let Z and A be distinct points and let a space-like line meet the light-cone at A at points P and Q. Cf. Figure 3. Consider a Euclidean plane containing this line but not the point A. The configuration in this plane is that which describes the Steiner power of Z with respect to the circle A. See Figure 5a where ZP.ZQ = (tangential distance Z to A)^2.

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[b] The cycles of Figure 1 have tangential distances in the ratio 3:4:5, and Figure 4 is a true perspective rendition of the corresponding configuration in M^3. The determination of the cycles in these figures and the handling of conics in Euclidean space made use of Lie's higher sphere geometry. Actual calculations were done on a pocket calculator and Figure 3 was drawn by hand. The use of Lie geometry in graphics problems will be the topic of a future paper by the authors.

[c] If one could observe, from a moving inertial frame, the circular wave fronts emitted by three pulses of light originating from sources at the vertices of a right triangle, one would "see" the configuration of the theorem of Pythagoras-Laguerre.

7.2. Let the assumptions be as in 7.1, except that the Euclidean plane contains none of the points Z, P, Q, A . The configuration in this plane is a Laguerre transformation of 7.1. One has $(\overrightarrow{ZP} | \overrightarrow{ZQ}) = (\overrightarrow{ZA} | \overrightarrow{ZA})$. See 4.6 and Figure 5b.

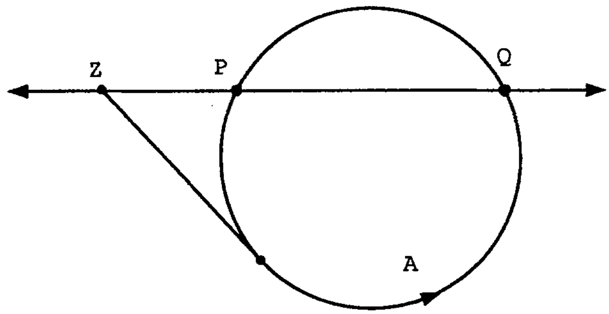

Figure 5a. Steiner power

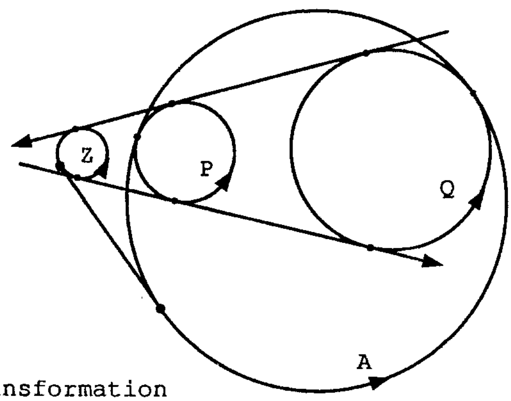

Figure 5b. Laguerre transformation of Steiner power

7.3. A Laguerre transformation applied to the configuration of Ptolomy's Theorem on cyclic quadrilaterals yields: If four cycles touch two additional cycles which have negative separation, then $d_{12}d_{34} + d_{23}d_{41} = d_{13}d_{24}$, where d_{ij} is the tangential distance between the i^{th} and j^{th} cycles. A version of such a theorem, requiring only that four circles be externally tangent to one additional circle, was proved by J. Casey in 1881. [2, Prop.10, p.103].

8. EXAMPLES IN THE MINKOWSKI PLANE

The development of Laguerre geometry for the Minkowski plane is completely analogous to that of the Euclidean plane.

8.1. The assumptions are as in 7.1, except that the plane is a Minkowski plane containing the points Z,P,Q but not the point A . The configuration in the Laguerre geometry of this plane, that is, of oriented Minkowski circles, is that which describes the Steiner power of Z with respect to the cycle A . See Figure 6a.

8.2. The assumptions are as in 8.1, except that the Minkowski plane contains none of the points Z, P, Q, A . The configuration in this plane is a Laguerre transformation of 8.1. See Figure 6b. This the Minkowski analog of 7.2.

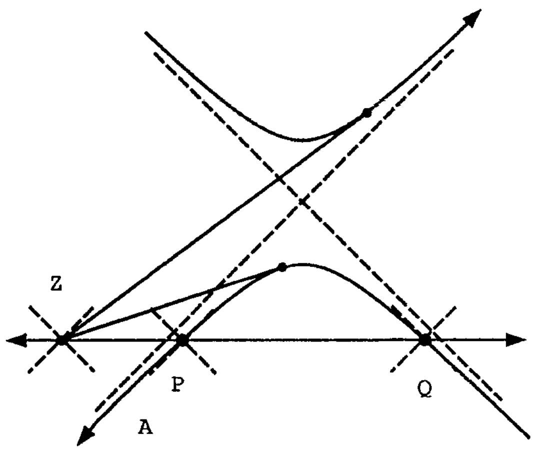

Figure 6a. Steiner power in the Minkowski plane

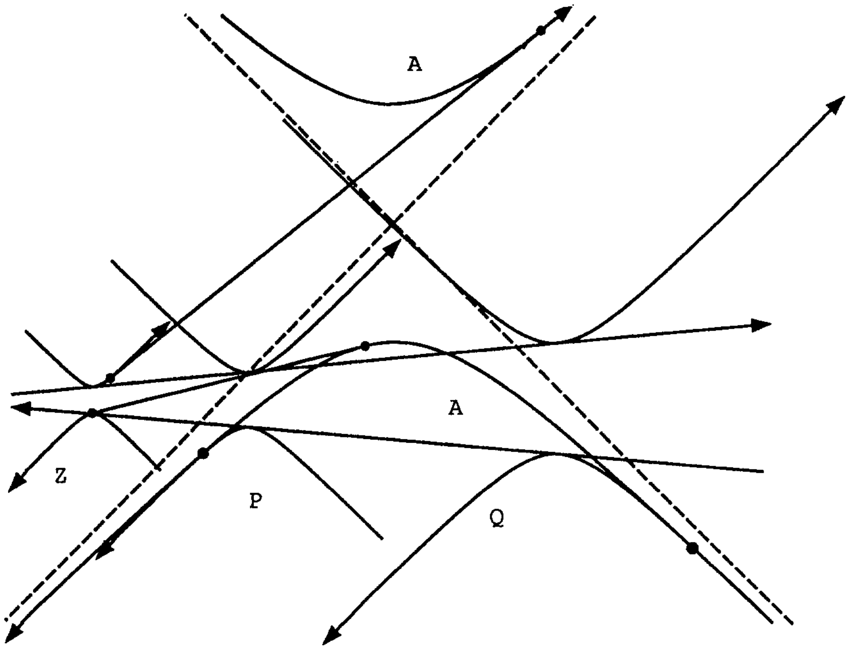

Figure 6b. Laguerre transformation of Steiner power in the Minkowski plane

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