The Apollonius Contact Problem and Representational Geometry

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Dedicated to Helmut Röhrl on the occasion of his seventieth birthday.

Abstract. The classical Apollonius contact problem, that of finding all circles tangent to three given circles, can be set in the context of representational geometry: circles correspond to points in the exterior of a nonruled quadric in projective space. Here we give new descriptions and proofs, using this representation, of the classical Gergonne construction of the solutions, and of the more recent observation that the Apollonius problem can never have exactly seven or exactly one solution.

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0. Introduction

The classical Apollonius contact problem in the Euclidean plane consists of finding those circles which are tangent to three given circles. During the XIX-th century it was usually considered as a construction problem. The most celebrated solution was that given by J. D. Gergonne [15, pp. 289–303] in the second decade of that century. This construction finds its place in almost every treatise on higher geometry: Johnson [19, pp. 119–121], Coolidge [5, pp. 170–172], Dörrie [7, pp. 159–160]. A very beautiful geometric proof of the key facts of this construction, algebraic and with a modern flavor, is provided by Salmon.

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A proof with a nice blend of the algebra and geometry is found in Graustein [16, pp. 375–376].

Until this century, little thought was given to enumerating the possible configurations which are solutions to the Apollonius problem. This enumeration is unwieldy at best. Since it is customary to state the Apollonius problem in terms of three given “things” – each thing being a circle, a line, or a point – and to seek such things as solutions, the many cases of the enumeration hinge upon the kind and the relative positions of the three given things. This complicated problem was first addressed by R. F. Muirhead in 1896 [22].

In spite of the difficulties, from such an enumeration one can note that there is no configuration of three circles for which a solution consists of exactly seven or exactly one circle (or line or point). This was first observed by D. Pedoe in 1970 [25] who used inversion to obtain a traditional proof.


Toward the end of that same century, “representational geometry” emerged and circles were represented by points of another space.

According to Bieberbach [3, p. 46], the transfer of one geometry to another dates back to the “Übertragungsprinzip” of O. Hesse in 1866 [17], but one must not overlook the “transfer” due to G. Darboux in 1873 [6].

One of the earliest such representations to treat the Apollonius problem was the “cyclographic” of W. Fiedler in 1882 [10], in which the center and radius of a circle furnish the coordinates of a point in three dimensional space. This is described in in the treatise of Coolidge [5, pp. 183–186].

In 1872 F. Klein represented circles by points exterior to a certain quadric in projective space [20, pp. 193–195] and [21] Bd. I, XXVII, § 6 (p. 475). (An affine version of the quadric has been used by Pedoe in 1937 [24].) Now problems like that of Apollonius could be naturally treated from the viewpoint of the projective geometry of a single quadric. See Fillmore and Paluszny [12].


The ideas of such representations of geometries played a central rôle in the geometries of Klein and S. Lie [20].

In this paper we set up the representation of inversive geometry in the spirit of Klein and use it to:

• exhibit a very simple construction, carried out using polarities in projective space, which lies at the heart of Gergonne’s construction. This not only clarifies the construction, but reveals a natural dual of the construction in the space of lines of the Euclidean plane.

• give a new proof of the fact that the Apollonius problem never has exactly seven or exactly one solution.
We begin with the Gergonne construction. It historically precedes enumeration and is the more interesting.

The methods used here extend to constructions in metric affine spaces of higher dimensions and other signatures. For the latter, see Fillmore and Paluszny [13] and the references cited there.

1. The classical Gergonne construction

Given three circles with non-collinear centers, the eight solutions (counting multiplicities) of the Apollonius problem can be paired by requiring that the radical center of the three circles is one of the two centers of similitude of a pair. The radical axis of two such solution circles is then one of the four lines of similitude of the given circles. The Gergonne construction reverses this to determine the points of contact that each one of the solution pairs will have with each of the given circles.

More specifically, Gergonne’s solution to the Apollonius problem consists of the following steps, as illustrated by Figure 1.

- Denoted by $A$, $B$, and $C$ three circles$^3$, in the Euclidean plane, whose centers are not collinear. Each pair of these circles has a radical axis and the three radical axes are concurrent in a point $o$ known as the radical center of the three circles.
- Let $E_{bc}$ and $I_{bc}$ denote the exterior and interior centers of similitude of circles $B$ and $C$. Likewise for circles $C$ and $A$ and for circles $A$ and $B$. By the theorem of d’Alembert, these six points lie by triples on four lines called the lines (of centers) of similitude: \{$E_{bc}, E_{ca}, E_{ab}$\}, \{$E_{bc}, I_{ca}, I_{ab}$\}, \{$I_{bc}, E_{ca}, I_{ab}$\}, \{$I_{bc}, I_{ca}, E_{ab}$\}.
- Fix one of the lines of similitude and call it $L$. Let the point $p_a$ be the pole of the line $L$ with respect to the given circle $A$. The line $o p_a$ then meets $A$ exactly at the points where a pair of solutions touch $A$.
- Likewise, using the same line of similitude $L$, obtain points on $B$ and $C$ where these same two solutions touch $B$ and $C$. We will say that this pair of solutions “corresponds” to $L$.
- For each line of similitude $L$ one so determines the points where the corresponding pair of solutions meet $A$, $B$, and $C$. The four choices for $L$ then yield the possible solutions to the Apollonius problem. There are at most eight.

It will be convenient to fix one of the given circles and one of the lines of similitude and refer to the third step above as the “Key Step” of the classical Gergonne construction. It is our goal to elucidate this Key Step by means of representational geometry.

2. Preliminaries

Throughout this paper, we will denote join by juxtaposition and intersection by a raised dot. For example, $A b \cdot \gamma$ denotes the (line of) intersection of the plane determined by the point $A$ and the line $b$ with the plane $\gamma$.

$^3$ The significance of the underscore and the choice of names will become evident in Sections 3 and 4.
2.1 Circles of antisimilitude. Given any two circles, the fixed point of a homothety taking one circle to the other is called a center of similitude of the two circles. A center of similitude is called exterior or interior according as the homothety has positive or negative scale factor. In general, two circles have one exterior and one interior center of similitude.

2.1.1 Theorem of d’Alembert and Monge: The six centers of similitude of the pairs of circles formed from three circles having noncollinear centers belong by threes to four lines.

For a proof, see [7, p.155], [16, pp.360–361], [18, pp.136–137].

The six centers of similitude are the vertices of a complete quadrilateral whose diagonal triangle is determined by the centers of the given circles.
The four sides of the quadrilateral are known as the axes of similitude of the three circles. One axis contains three external centers of similitude, each of the remaining three contains one external and two internal centers of similitude.

Inversion with center $M$ and power $p$ is the map sending $X$ to $X'$, where $M$, $X$, and $X'$ are collinear and $MX.MX' = p$ (product of signed distances). If the power $p$ is positive, the fixed points are the circle with center $M$ and radius $\sqrt{p}$ and the map is known as inversion in that circle. If the power $p$ is negative, the map has no fixed points; it is inversion in a circle of radius $\sqrt{-p}$ followed by a half-turn. It is customary to say that the circle of inversion has "imaginary radius" — to distinguish this from inversions in circles (which have a real radius). Either kind of inversion sends circles and straight lines to circles and straight lines. In particular, a circle which does not pass through the center of inversion is sent to another such circle.

There are, in general, exactly two inversions which interchange two given circles [16, p.368]. The centers of these inversions are the centers of similitude of the two circles. The circles of inversion are called the circles of antisimilitude (or mid-circles) of the two circles. A circle of antisimilitude is called exterior or interior according as the center of the inversion is an exterior or interior center of similitude. Important are:

- The external circle of antisimilitude of two circles has a real radius if and only if the external center of similitude is outside the given circles.
- The internal circle of antisimilitude of two circles has a real radius if and only if the internal center of similitude is inside the given circles.
- The two circles of antisimilitude are coaxal with the given circles.

Analogous to and derived from 2.1.1 is:

**2.1.2** The six circles of antisimilitude of the pairs of circles formed from three circles having noncollinear centers belong by threes to four coaxal systems. The lines of centers of the four systems are the axes of similitude of the given circles.

See Graustein [16, p.369] and Steiner [29, p.145].

Note that circles of imaginary radius can be included in a coaxal system by considering the family of inversions. See also Section 3.

**2.2** Polarities. Our description of Gergonne's construction will make use of a polarity with respect to a degenerate quadric. Many of the well known facts about polarities with respect to a nondegenerate quadric continue to hold when the quadric is degenerate [26] Vol. II Geometry, Ch. 10, 4.5 (pp.411–412). Since these are seldom used, we summarize here what will be needed.

Let $P^3$ denote real projective space.

A cone $\psi$ in $P^3$ is determined by its vertex $V$ and generators which are lines joining $V$ to points of the nondegenerate conic $\psi \cdot \pi$ in any (fixed) plane $\pi$ of $P^3$ not meeting $V$. A cone is a degenerate quadric and familiar notions regarding quadrics carry over to cones.

Two points $X$ and $Y$ of $P^3$ are called conjugate with respect to $\psi$ if the line $XY$ does not pass through $V$ and meets $\psi$ in two points which separate $X$ and $Y$ harmonically. If $P$ is a point distinct from $V$, all points $X$ conjugate to $P$ lie on a plane of $P^3$ which we denote by $P^b$ and is called the polar plane of $P$ with respect to $\psi$. It passes through the
vertex $V$ of the cone. The transformation which sends the point $P$ to the plane $P^b$ is the polarity with respect to $\psi$. It is not invertible.

If $g$ is a line of $\mathbb{P}^3$ not meeting $V$, then the intersection of all the polar planes of points of $g$ is a line through $V$ which we denote by $g^b$. (If $g$ is a line of $\mathbb{P}^3$ passing through $V$, then $g^b$ is the polar plane of any point of $g - \{V\}$.)

Note: If $\pi$ is a (fixed) plane of $\mathbb{P}^3$ not passing through $V$, then, in the plane $\pi$, the line $P^b \cdot \pi$ is the polar of the point $VP \cdot \pi$ with respect to the nondegenerate conic $\psi \cdot \pi$. The polarity $\hat{b}$ induces in $\pi$ the polarity with respect to $\psi \cdot \pi$.

2.3 A special construction. Let $\Psi$ be a nondegenerate quadric in $\mathbb{P}^3$ which is not ruled. The exterior $\Psi^+$ of the quadric $\Psi$ consists of points through which there passes a line of $\mathbb{P}^3$ which does not meet $\Psi$.

We denote by $P^\perp$ the polar plane of the point $P$ with respect to $\Psi$. The polar of a line $g$ which joins two points is the line $g^\perp$ which is the intersection of the polar planes of the two points. The polarity with respect to the quadric $\Psi$ is an involution.

Lines tangent to $\Psi$ from a point $V$ in the exterior $\Psi^+$ are the generators of the tangent cone to $\Psi$ from $V$. We denote this cone by $\psi_V$. This cone is tangent to $\Psi$ at points of the conic $\Psi \cdot V^\perp$.

Let $b$ be the polarity with respect to the cone $\psi_V$. Then the composition of $b$ followed by the polarity $\perp$ with respect to $\Psi$ is just the projection of $\mathbb{P}^3 - \{V\}$ to the polar plane $V^\perp$:

$$(P^\perp)^\perp = (\text{line } VP) \cdot V^\perp.$$  

Applied to points determining a line $g$ not through $V$, this gives

$$(g^b)^\perp = (\text{plane } V g) \cdot V^\perp.$$  

This projection, which sends the line $g$ not meeting $V$ to the line $(g^b)^\perp$ of $V^\perp$, will be seen to be the heart of the Key Step of Gergonne’s construction. See Section 4.

3. The representation of circles

3.1 Let $E^2$ be a plane and $S^2$ a sphere in Euclidean space. Choose a “north pole” on $S^2$: the tangent plane to $S^2$ at this point is to be parallel to but distinct from $E^2$. By means of familiar stereographic projection, circles and lines of $E^2$ correspond to circles on $S^2$.

Note that two circles of $E^2$ are tangent, a circle and a line are tangent, or two lines are parallel exactly the corresponding circles on $S^2$ are tangent.

Continuing: Each circle of $S^2$ whose plane does not pass through the center of the sphere corresponds to a unique point exterior to $S^2$: the pole (with respect to the sphere) of the plane of the circle.

In order to represent all circles on $S^2$, extend Euclidean space to projective space $\mathbb{P}^3$ and note that $S^2$ with its north pole is a nonruled quadric with a distinguished point chosen

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4 Mnemonic for $\perp$ and $b$: A nonsingular polarity is a ‘natural’ for duality, a singular polarity makes configurations ‘flat’.
on it. Points of the exterior of this quadric now yield all circles of $S^2$ as the intersection of the quadric with polar planes of points in the exterior.

Note that two circles of $S^2$ are tangent exactly when the line joining their poles is tangent to $S^2$.

We will work exclusively in projective space $P^3$ with a nonruled quadric $\Psi$ and a point $N$ chosen on it, guided by the fact that points in the exterior $\Psi^+$ of $\Psi$ represent circles and lines of a Euclidean plane — which we will continue to call $E^2$.

In order to refer easily to the configurations in the Euclidean plane, we will denote by $\mathcal{A}$ the circle or line of $E^2$ represented by the point $A$ of $\Psi^+$. Likewise, the image of the point $P$ in $\Psi - \{N\}$ under stereographic projection will be denoted $\mathcal{P}$. (It is a circle of radius zero.)

Now we have: Circles (or lines) $A$ and $B$ in $E^2$ are tangent exactly when the line $AB$ in $P^3$ is tangent to the quadric $\Psi$.

We will denote the polarity with respect to the quadric $\Psi$ by $\Psi$.

3.2 A point $P$ of $P^3$ not on $\Psi$ determines a harmonic homology: The image of a point $X$ is the point $X'$ which is the harmonic conjugate of $X$ with respect to $P$ and the point $PX \cdot P^3$ where the line $PX$ meets the polar plane of $P$.

• If $P$ is in the exterior $\Psi^+$ of $\Psi$, this involution corresponds in $E^2$ to inversion in the circle $\mathcal{P}$: the fixed points of the involution are the points of $\Psi$ which represent the points of the circle $\mathcal{P}$; the image of $X$ in $\Psi$ is the point of $\Psi$ which represents the point of $E^2$ obtained by inverting the point $X$ in the circle $\mathcal{P}$; the image of $A$ in $\Psi^+$ is the point of $\Psi^+$ which represents the circle of $E^2$ obtained by inverting the circle $\mathcal{A}$ in the circle $\mathcal{P}$.

• If $P$ is in the “interior” $P^3 - \{\Psi \cup \Psi^+\}$, this involution corresponds in $E^2$ to inversion in a circle followed by a half-turn about the center of that circle. This involution has no fixed points and is said to be inversion in a circle of “imaginary radius”. The images of points in $\Psi$ and $\Psi^+$ represent points and circles of $E^2$ obtained by inversion in a circle followed by a half-turn.

We remark in passing: Inversions in circles generate the two-component inversive group of the Euclidean plane $E^2$. The harmonic homologies generate the projective orthogonal group $PO(1, 3)$ (of signature $-+++$). The above remarks establish the isomorphism of the inversive group with $PO(1, 3)$. For elaboration of this, see Alexander [1], Fillmore [11], and Wilker [30].

3.3 We have the following important link:

**Proposition.** Any pair of points $A$ and $B$ for which the line $AB$ is not tangent to $\Psi$ determine a unique pair of points $K$ and $K'$ on $AB$ which are conjugate with respect to $\Psi$ and separate $A$ and $B$ harmonically. If, in addition, the points $A$ and $B$ are in $\Psi^+$, then the circles $K$ and $K'$ are the circles of antismimilitude of the circles $A$ and $B$.

**Proof** Suppose that the line $AB$ meets $\Psi$ in two points. Choose a projective parameter on $AB$ so that these two points are given by 0 and $\infty$, and $A, B, K, K'$ are given respectively by $a, b, k, k'$. For $K$ and $K'$ to be conjugate with respect to $\Psi$, we must have $k + k' = 0$. For $K$ and $K'$ to separate $A$ and $B$ harmonically, we must have $2kk' - (a + b)(k + k') + 2ab = 0$. This yields that $k$ and $k'$ are $\pm \sqrt{ab}$. 
If we permit the parameters to be complex, the same argument holds when the line $AB$ does not meet $\Psi$.

On $AB$ the harmonic homology determined by the point with parameter $p$ is given by $x \mapsto p^2/x$. When $p^2 = ab$, the parameters $a$ and $b$ are interchanged. Thus, inversion in $K$ or $K'$ interchanges the circles $A'$ and $B'$. QED

The points $K$ and $K'$ will be called the antismimilitude points of $A$ and $B$. At least one of the points $K$ and $K'$ must lie in the exterior of $\Psi^+$ and both may. (If $A$ and $B$ meet, then both $K$ and $K'$ have real radii. If $A$ and $B$ do not meet, then only one of $K$ and $K'$ has real radius.)

3.4 The representation of circles and lines by points in $\Psi^+$ yields a simple interpretation of 2.1.2:

**Corollary.** The six antismimilitude points formed from three given noncollinear points of $\Psi^+$ belong by threes to four lines of a complete quadrilateral in a plane in $\mathbb{P}^3$. The diagonal triangle of this quadrilateral has as vertices the given points.

The configuration of 2.1.1 may be obtained from this corollary by means of projection with center $N$ from $\mathbb{P}^3 - \{N\}$ to $E^2$.

The four lines of the corollary will be called the antismimilitude lines for the three given points.

3.5 We now turn to the first steps for the Apollonius contact problem: Given three noncoaxal circles or lines in $E^2$ find all circles and lines tangent to the three given. We will routinely:

- denote the points of $\Psi^+$ which represent the three given circles or lines of this problem by $A$, $B$, $C$.
- denote the four antismimilitude lines for $A$, $B$, $C$ by $\ell_1, \ell_2, \ell_3, \ell_4$.
- denote the pole of the plane of the noncollinear points $A$, $B$, $C$ by $Q$.
- omit "or line" in interpretations in $E^2$.

If $V$ is a point in $\Psi^+$, let $\psi_V$ denote the tangent cone to $\Psi$ with vertex at $V$. The points at which $\psi_V$ touches $\Psi$ lie on the intersection if $\Psi$ and the polar plane $V^*$. Points of $\psi_V$ represent circles tangent to the circle $V$. Thus, if the three given circles are represented by the three noncollinear points $A$, $B$, $C$ in $\Psi^+$, the solution of the Apollonius contact problem is obtained as the circles represented by the intersection of three cones: $\psi_A \cdot \psi_B \cdot \psi_C$.

3.6 The intersection of two cones in $\mathbb{P}^3$ is, in general, a quartic curve. It case that the two cones are tangent to a quadric, the situation is simpler.

**Theorem.** Let $V$ and $W$ be distinct points in $\Psi^+$ and $K$ and $K'$ their antismimilitude points. Then, the intersection of cones $\psi_V \cdot \psi_W$ consists of the the two conics $\psi_V \cdot K^3 = \psi_W \cdot K^3$ and $\psi_V \cdot K'^3 = \psi_W \cdot K'^3$ lying in the polar planes (with respect to $\Psi$) of $K$ and $K'$.

**Proof:** In any plane $\pi$ containing $V$ and $W$, the two lines of $\psi_V \cdot \pi$, together with the two lines of $\psi_W \cdot \pi$, are four lines of the conic envelope of $\Psi \cdot \pi$. The points $V$, $W$, and two additional points are on a side of a complete quadrilateral. These two additional points
separate $V$ and $W$ harmonically. The diagonal triangle of this complete quadrilateral is self polar (by the dual of [28] Ch. V §4, Th.4, p.109), the two additional points are conjugate with respect to $\Psi$. Hence, these two points are the antismilitude points $K$ and $K'$. The points of $\psi_V \cdot \psi_W$ which lie in $\pi$ are the points in which the lines of $\psi_V \cdot \pi$ and $\psi_W \cdot \pi$ meet. These points lie on $K^a$ and $K'^a$. QED

**Consequence:** The intersection $\psi_A \cdot \psi_B \cdot \psi_C$ of three cones with noncollinear vertices is obtained by intersection of one of the cones with the polar lines of the four antismilitude lines of the three vertices $A, B, C$:

$$\psi_A \cdot \psi_B \cdot \psi_C = \psi_A \cdot \ell_1^a \cup \psi_A \cdot \ell_2^a \cup \psi_A \cdot \ell_3^a \cup \psi_A \cdot \ell_4^a.$$ 

**Proof:** Two applications of the lemma yields:

$$(\psi_A \cdot \psi_B) \cdot \psi_C = (\psi_A \cdot K_{ab}^a \cup \psi_A \cdot K_{ab}^{a'}) \cdot \psi_C$$
$$= (\psi_A \cdot \psi_C) \cdot K_{ab}^a \cup (\psi_A \cdot \psi_C) \cdot K_{ab}^{a'}$$
$$= (\psi_A \cdot K_{ca}^a \cup \psi_A \cdot K_{ca}^{a'}) \cdot K_{ab} \cup \text{(two more terms)}$$
$$= \psi_A \cdot K_{ca}^a \cdot K_{ab}^a \cup \text{(three more terms)}$$
$$= \psi_A \cdot (\text{line } K_{ca}K_{ab})^a \cup \text{(three more terms)}$$

QED

See also [12] and [13].

Note that the lines $\ell_1^a, ..., \ell_4^a$ are concurrent at the pole $Q$ of plane $ABC$, since each of these lines lies in the plane $Q^a$.

3.7 We can create a “dictionary” which relates the projective and Euclidean interpretations which we need for the Gergonne construction. Recall that $A$ denotes the circle or line of $E^2$ represented by the point $A$ in $\Psi^+$.

### Projective Space $\Psi^+$
- Point of $\Psi - \{N\}$
- Point of $\Psi^+ - N^a$
- Point of $N^a - \{N\}$
- Line $XA$ tangent to $\Psi$ at $P$
  - $X$ in $N^a - \{N\}$
  - $A$ in $\Psi^+ - N^a$
- Line not lying in $N^a$ and meeting $\Psi$ in two points, tangent to $\Psi$, not meeting $\Psi$.

### Euclidean Plane $E^2$
- Point
- Circle
- Line
- Line $X$ tangent to circle $A$ at point $P$
- Pencil of non-intersecting, tangent, intersecting circles.
Line lying in $N^3$
meeting $N$,
not meeting $N$.

Points in which a line
meets $\Psi$.
Second intersection of the
line from $A$ to $N$ with $\Psi$.
Points $A$ and $B$ conjugate
with respect to $\Psi$.
Plane
intersecting
not meeting
tangent to
(except $N^3$)
the quadric $\Psi$.
Point of antisimilitude
for two points.
$(line) \cdot N^3$  
Polar line with respect to $\Psi$.
$(line)^3 \cdot N^3$  
Pole of a plane
meeting $\Psi$.

Pole of a plane
tangent to $\Psi$.

And for: $\bullet$ $A, B, C$ three noncollinear points of $\Psi^+$
$\bullet$ $\ell$ an antisimilitude line of $A, B, C$ in $P^3$

Antisimilitude points
$K_{bc}, K_{ca}, K_{ab}$
lying on the line $\ell$.

$\ell^3 \cdot N^3$

Pencil of
parallel lines,
concurrent lines,
or the point
of concurrency.
Poncelet points
of a pencil of circles.
Center of the circle $A$.
Orthogonal circles
$A$ and $B$.
Circles of a
hyperbolic
elliptic
parabolic
(all lines)
bundle.
Circle of antisimilitude
of two circles.
Radical axis of a pencil.
Conjugate pencil.
Line of centers of a pencil.
Circle orthogonal
to all circles
of a hyperbolic bundle.
Point common to all circles
of a parabolic bundle.

3.8 We now take representational geometry one step further: The plane $N^3$, which is
tangent to $\Psi$ at the “north pole” $N$, plays the rôle of a dual to the Euclidean plane $E^2$.
To establish this, we work in terms of $\Psi$.

In one direction:
$\bullet$ If $X$ is a point of $N^3 - \{N\}$, then $X = (\Psi - \{N\}) \cdot X^3$ is a circle of $\Psi$ (less the point $N$)
which we refer to as a line of $E^2$. 
• If \( g \) is a line of \( N^2 \) not passing through \( N \), then \( (\Psi - \{N\}) \cdot g^3 \) is a point of \( \Psi \) which we denote by \( g \) and refer to as the point of \( E^2 \) represented by \( g \).

Observe that the lines \( g \) and \( g^3 \) in \( P^3 \) represent a pencil of concurrent lines and its orthogonal pencil of concentric circles in \( E^2 \), respectively. The point \( g \) is the point of concurrency and the common centers of the circles for the respective pencils.

We will underscore a line of \( P^3 \) (to indicate the point of \( E^2 \) which it represents) only when the line lies in \( N^3 \).

In the other direction:
• If \( P \) is a point of \( \Psi - \{N\} \), then \( (NP)^3 = N^3 \cdot P^3 \) is the line of \( N^3 \) representing it.

And the familiar:
• A line of \( E^2 \) is a plane of \( P^3 \) passing through \( N \) represented by the point if \( N^3 \) which is the pole of that plane.

Observe: If \( X \) and \( Y \) are points of \( N^3 - \{N\} \), then \( XY = X \cdot Y \), that is, the line \( XY \) in \( N^3 \) represents the point of \( E^2 \) which is the intersection of the lines \( X \) and \( Y \).

And: If \( g \) and \( h \) are lines of \( N^3 \) (not passing through \( N \)), then \( g \cdot h = g \cdot h \), that is, the point \( g \cdot h \) in \( N^3 \) represents the line of \( E^2 \) joining the points \( g \) and \( h \).

3.9 We proceed to interpret, in the plane \( N^3 \), various geometric objects relevant to the Key Step for the Gergonne construction in the Euclidean plane \( E^2 \). For this purpose we make another, much shorter, dictionary using the first.

**Dual plane \( N^3 \) \hspace{1cm} \text{Euclidean Plane} \ E^2**

**Point \( X \) in \( N^3 - \{N\} \) \hspace{1cm} \text{Line} \ X**

**Line \( g \) in \( N^3 - \{N\} \) \hspace{1cm} \text{Point} \ g \) which is the common center of circles in a pencil of concentric circles - or - point circle meeting the lines of a pencil of concurrent lines**

**Line \( N^3 \cdot P^3 = (\text{line} \ NP)^3 \) \hspace{1cm} \text{Point} \ P**

**Line \( N^3 \cdot A^3 = (\text{line} \ NA)^3 \) \hspace{1cm} \text{Center of circle} \ A**

**Point \( (\text{line} \ AB) \cdot N^3 \) \hspace{1cm} \text{Radical axis of the circles} \ A \text{ and } B**

**Line \( Q^3 \cdot N^3 \) \hspace{1cm} \text{Radical center of any three circles represented by non-collinear points of the plane} Q^3.**

3.10 For \( A \) in \( \Psi^+ - N^3 \) let \( \phi_A \) denote the nondegenerate conic \( \psi_A \cdot N^3 \) in \( N^4 \).

**Proposition.** Let \( A \) in \( \Psi^+ - N^3 \). If, in \( N^3 \), the line \( g \) is tangent to \( \phi_A \) at the point \( X \), then, in \( E^2 \), the line \( X \) is tangent to the circle \( A \) at the point \( g \).

This is clear from the dictionaries in 3.7 and 3.9.
Theorem. Let $A$ in $\Psi^+ - N^1$. If two points are conjugate with respect to the conic $\phi_A$ in $N^3$, then the corresponding two lines are conjugate with respect to the circle $A$ in $E^2$.

Proof. Suppose that $U, V, X, Y$ are a harmonic tetrad of points lying on a line $g$ in $N^3$. Then the planes $U^g, V^g, X^g, Y^g$ in $P^3$ are coaxal - and a harmonic tetrad. The process of stereographic projection yields lines $U, V, X, Y$ which are concurrent at $g$ - and a harmonic tetrad. If $U$ and $V$ lie on $\phi_A$, so that $U$ and $V$ separate $X$ and $Y$ harmonically, then $U$ and $V$ are tangent to $A$, so that $U$ and $V$ separate $X$ and $Y$ harmonically. QED

A similar proof holds starting with two lines in $N^3$ and obtaining two points in $E^2$.

Consequently: The polarity with respect to the circle $A$ in $E^2$ transports to the polarity with respect to the conic $\phi_A$ in $N^3$. This correspondence is, in fact, a correlation.

4. The Gergonne construction

In this section we consider a simple construction in projective space $P^3$ and show it is equivalent to the Key Step of the Gergonne construction. This discussion is illustrated by Figure 2.

4.1 Let $N$ be the “north pole” of the quadric $\Psi$. The polarity with respect to $\Psi$ is denoted $\psi$ as before. Let $A, B, C$ be three noncollinear points in $\Psi^+$, none of which lie in $N^3$, and assume that the pole $Q$ of plane $ABC$ also does not lie on $N^3$. Choose and fix: One of the three given points, say $A$, and one of the four lines of antisimilitude, call it $\ell$.

In the setting of 2.2, take for $V$ the point $A$, for $g$ the polar $\ell^g$, and for $\pi$ the plane $N^g$. The degenerate polarity $b$ is then with respect to the cone $\psi_A$. Recall that this degenerate polarity induces on $N^g$ the polarity with respect to the nondegenerate conic $\phi_A = \psi_A \cdot N^g$.

4.2 Note that the point $L = \ell^g \cdot N^g$ represents the line of similitude $L$ on which lie the centers of the circles of antisimilitude represented by points on $\ell$.

The line $\ell^g$ joins the points $L$ and $Q$ (since $Q$ does not lie in $N^g$). Applying the degenerate polarity, we obtain $(\ell^g)^b = L^b \cdot Q^b$, so $(\ell^g)^b$ is the intersection of the two planes $L^b$ and $Q^b$ in $P^3$. If we intersect with $N^g$ we have

$$(\ell^g)^b \cdot N^g = L^b \cdot N^3 \cdot Q^b \cdot N^g$$

That is, the point $(\ell^g)^b \cdot N^g$, which we denote by $M$, is the intersection of the two lines $L^b \cdot N^g$ and $Q^b \cdot N^g$ in $N^3$.

This point $M$ is the basic construct of the Key Step. For: The line $L^b \cdot N^g$ is the polar of $L$ with respect to $\phi_A$, and $L$ is the line of similitude associated with $\ell^g$. Since the polarity on $N^g$ with respect to $\phi_A$ corresponds to the polarity in $E^2$ with respect to the circle $A$, the line $p_a = L^b \cdot N^g$ of $N^3$ represents the pole $p_a$ of the line of similitude $L$ with respect to $A$. See 3.9. And: $Q^b = Q^g$, since $A$ and $Q$ are conjugate with respect to $\Psi$. So, the line $o = Q^b \cdot N^g = Q^g \cdot N^3$ represents the radical center $o$ of the circles $A, B$, and $C$. Again see 3.9. Hence, $M$ in $N^g$ represents the Euclidean line joining the radical center with the pole with respect to $A$ of the line of similitude $L$: $M = o \cdot p_a = o p_a$. 

4.3 We can now show that, in the context of this representation, the points (if any) at which the line $M$ meets the circle $A$ are precisely those where the pair of solutions (counting multiplicities) lying on $\ell^b$ touch the circle $A$.

First observe that $(\ell^b)^h$ is the line joining $M$ and $A$. ($A$ and $M$ are distinct since $A$ does not lie on $N^b$.) For, $M$ lies on $(\ell^b)^h$ by definition and the polar, with respect to $\psi_A$, of any line is a line passing through $A$. We now interpret

$$(\ell^b)^h \cdot \Psi = (\text{line } MA)^h \cdot \Psi.$$  

For the left side: Since $(\ell^b)^h$ is projection to $A^h$ from $A$, a point (if any) $S'$ of $((\ell^b)^h)^h \cdot \Psi$ is the image of a point $S$ on $\ell^h \cdot \psi_A$. Since the line $ASS'$ lies in $\psi_A$, $S'$ in $\Psi$ corresponds to the point at which the circle $S$ is tangent to the circle $A$. See Figure 2.

For the right side: The points (if any) at which (line $MA)^h$ meets $\Psi$ are the Poncelet points of that pencil and hence the common points of the conjugate pencil corresponding to (line $MA$), that is, the points where the line $M$ meets the circle $A$ in the Euclidean plane.
Thus, the equality shows that the circles which are solutions represented by points on \( \ell^2 \) touch \( A \) are the points where the line \( M \) meets \( A \).

Finally, we remark that the Key Step of the Gergonne construction fails, if the circles \( A, B, \) and \( C \), have collinear centers. See Coolidge [5, p.172]. This corresponds to the fact that \( M = (\ell^4)^* \cdot N^4 \) cannot be obtained as the intersection of the two lines in \( N^4 \) representing the pole of the simililitude line with respect to \( A \) and the radical center. For, \( Q \) and \( \ell^4 \cdot N^4 \) coincide.

5. Seven and one solutions

5.1 We will need the following additional observations regarding the antissimilitude lines \( \ell_1, \ell_2, \ell_3, \ell_4 \).

From 3.4 and 3.5:
- The four antissimilitude lines are distinct, or equivalently, no four distinct antissimilitude points are collinear.
- Any two antissimilitude lines meet at an antissimilitude point.

Also note that none of the lines \( \ell_1^\perp, ..., \ell_4^\perp \) can lie in – or be a generator of – any of the cones \( \psi_A, \psi_B, \psi_C \). Since \( \psi_A \cdot \ell^\perp = \psi_B \cdot \ell^\perp = \psi_C \cdot \ell^\perp \) for any one of the antissimilitude lines \( \ell \), if \( \ell^\perp \) were to be a generator of one cone, it would be a generator of all three cones. This would contradict the noncollinearity of \( A, B, C \).

5.1.1 Consequence: Each line \( \ell^\perp \) contributes at most two solutions to the Apollonius problem. So there are at most eight solutions, grouped in four pairs, each pair lying on one of the lines \( \ell^\perp \). The pole \( Q \) of the plane \( Q^\perp \), on which lie \( A, B, C \) is the point of concurrence of the lines \( \ell^\perp \). If the plane \( Q^\perp \) intersects \( \Psi \), then \( Q \) is the circle orthogonal to the circles \( A, B, C \). Inversion in \( Q \) interchanges the solutions given by any one of the lines \( \ell^\perp \). In particular if \( \ell^\perp \) is tangent to the cones, then the solution it contributes is orthogonal to \( Q \).

In the case that \( A, B, C \) are noncollinear points lying in \( \Psi^+ \) or \( \Psi \), so that some represent points of \( E^2 \), we may still consider the Apollonius problem. We now obtain fewer than four antissimilitude lines. For example, if \( A \) lies in \( \Psi^+ \) and \( B \) and \( C \) lie in \( \Psi \), then the antissimilitude points on \( BC \) are not defined. Nevertheless, the solutions to the Apollonius problem are \( \psi_A \cdot \psi_B \cdot \psi_C \) and, since \( \psi_B = B^\perp \) and \( \psi_C = C^\perp \), we obtain that the solutions are the intersection of the line \( B^\perp \cdot C^\perp = (BC)^\perp \) with \( \psi_A \).

5.1.2 Another consequence: The line which is the intersection of the polars of two antissimilitude points on different of the lines \( BC, CA, AB \) meets each of the cones \( \psi_A, \psi_B, \psi_C \) in the same points.

Proof. We have \( \psi_A \cdot K_{ab} \cdot K'_{ca} = \psi_B \cdot K_{ab} \cdot K'_{ca} \) and \( \psi_A \cdot K_{ca} \cdot K'_{ba} = \psi_C \cdot K_{ca} \cdot K'_{ba} \). QED

5.2 Let \( A, B, C \) be three distinct points of \( \Psi^+ \cup \Psi \) in \( P^3 \). Again, we do not distinguish between circles and lines of \( E^2 \) which the points of \( \Psi^+ \) represent, but we do distinguish points of \( \Psi \) as representing “point circles” or points of \( E^2 \). If \( A \) is a point of \( \Psi \), then the cone \( \psi_A \) is to be understood as the polar plane \( A^\perp \) which is tangent to \( \Psi \) at the point \( A \).
It is still the case that the solutions to the Apollonius problem are represented by points obtained as the intersection $\psi_A \cdot \psi_B \cdot \psi_C$.

We first dispose of some special cases which are easily determined - even by traditional methods.

5.3 Concerning the possibility of exactly seven solutions:

- If the points $A, B, C$ are collinear, that is, the circles $A, B, C$ are in a pencil, then there are either no solutions or infinitely many solutions.
- If one (or more) of $A, B, C$ is in $\Psi$, then there are at most four solutions.
- If one (or more) of the lines $BC, CA, AB$ are tangent to $\Psi$, then there are at most six solutions.

Hence, if exactly seven solutions are to occur, the points $A, B, C$ must all lie in $\Psi^+$ and be noncollinear, and no side of triangle $ABC$ can be tangent to $\Psi$.

5.4 Concerning the possibility of exactly one solution:

- Suppose that all three of $A, B, C$ lie on $\Psi$. Then $\psi_A \cdot \psi_B \cdot \psi_C$ is the intersection of three tangent planes $A^\perp \cdot B^\perp \cdot C^\perp$. This is the point in $\Psi^+$ which is the pole of the plane $ABC$. (In $E^2$ this is saying that three points determine a unique circle!)
- Suppose that $A$ and $B$ lie in $\Psi$ and $C$ lies in $\Psi^+$. If the line $BC$ is tangent to $\Psi$ at $B$, then $\psi_B \cdot \psi_C = B^\perp \cdot \psi_C$ is the line $BC$. This line meets the plane $A^\perp$ in a single point.
- Suppose that only $A$ lies in $\Psi$ and $B$ and $C$ lie in $\Psi^+$. If line $BC$ is tangent to $\Psi$, then $\psi_B \cdot \psi_C$ is the union of a conic and a line. If the plane $A^\perp$ meets the line but not the conic, then the Apollonius problem has a unique solution.

Thus the interesting question is: Can the Apollonius problem have exactly one solution when the three givens are circles (or lines) but none are points? The case of $A, B, C$ collinear is treated as in 5.3.

5.5 Based on 5.3 and 5.4, our standing assumptions will be:

(S1) The points $A, B, C$ lie in $\Psi^+$ and are not collinear.

(S2) None of the lines $BC, CA, AB$ is tangent to $\Psi$.

Another will be added later.

We begin by studying the relation between the position of an antisimilitude lines $\ell$ with respect to $A, B, C$, and the number of solutions lying on the polar line $\ell^\perp$.

5.5.1 Proposition. If $\ell^\perp$ does not pass through $A$, then $\ell^\perp$ contributes two, one, or no solutions to the Apollonius problem according as $\ell^\perp$ lies in $\Psi^+$, in $\Psi$, or in $\mathbb{P}^3 - (\Psi^+ \cup \Psi)$.

Proof. The following statements are equivalent: $\ell^\perp$ contributes two solutions; $\psi_A \cdot \ell^\perp$ consists of two points; the plane of $A$ and $\ell^\perp$ meets $\Psi$ (in more than a point); and the pole $A^\perp \cdot \ell$ of the plane of $A$ and $\ell^\perp$ lies in $\Psi^+$. This proves the first assertion. For the second assertion, again observe that equivalent are: $\ell^\perp$ contributes one solution; $\psi_A \cdot \ell^\perp$ consists of one point; the plane of $A$ and $\ell^\perp$ is tangent to $\Psi$; and the pole $A^\perp \cdot \ell$ of the plane of $A$ and $\ell^\perp$ lies in $\Psi$. The remaining case is now clear. QED
5.5.2 Proposition. Under the assumptions (S1) and (S2): If the plane ABC is tangent to \( \Psi \), then there are exactly five solutions to the Apollonius problem.

Proof. The point of tangency of the plane \( ABC \) with \( \Psi \) is its pole \( Q \). For the purpose of applying 5.5.1, fix a choice of vertex \( A \). Let \( \ell \) be any antismilitude line. Claim: \( \ell \) does not pass through \( Q \). Assume, for contradiction, that \( Q \) lies on \( \ell \). Since \( Q \) lies on none of the lines \( BC, CA, AB \), the point \( Q \) together with the three antismilitude points \( \{ K \} \) on \( \ell \) are four collinear points lying on the line \( \ell \) tangent to \( \Psi \) at \( Q \). The polars of these four points are four coaxal planes with axis \( \ell \perp \) tangent to \( \Psi \) at \( Q \). Since the antismilitude point \( K' \) on \( \ell \) conjugate to \( K \) lies on the plane \( K \perp \), the three points \( \{ K' \} \) would be collinear on \( \ell \perp \). But these three points are not one of the four antismilitude lines. Thus, \( \ell \cdot A \perp \) is a point distinct from \( Q \). By 5.5.1, each such antismilitude line \( \ell \) contributes two solutions, one of which is \( Q \). From the four antismilitude lines we thus obtain five solutions. \( \text{QED} \)

Consequence: If the Apollonius contact problem is to have exactly seven or one solution, the plane \( ABC \) must meet \( \Psi \) in a conic (not a point and nondegenerate).

We will add this as another standing assumption:

(S3) The plane \( ABC \) meets \( \Psi \) in a conic.

Note that this is equivalent to the pole \( Q \) of the lying in \( \Psi^+ \).

5.6 We finally arrive at:

Theorem. The Apollonius contact problem can never have exactly seven solutions and, except when one or more of the gives is a point, cannot have exactly one solution.

Proof. We assume (S1), (S2), and (S3).

If there are to be seven or one solutions, it must be the case that one of the antismilitude line, say \( \ell \), has polar \( \ell \perp \) which meets each cone \( \psi_A, \psi_B, \psi_C \) in a single point.

We first note that \( \ell \perp \) can pass through none of the points \( A, B, C \). For, if \( \ell \perp \) were to meet \( \psi_A \) at \( A \), then \( A \) would represent a solution to the Apollonius problem and consequently lie on \( \psi_B \) (and \( \psi_C \)). But then the line \( AB \) would be tangent to \( \Psi \), contrary to (S2).

For the moment, focus on the line \( \ell \) and one cone \( \psi_A \). The line \( \ell \perp \) meets \( \psi_A \) in a single point other than the vertex. Since \( \ell \perp \) is tangent to \( \psi_A \), the plane of \( \ell \perp \) and is tangent to \( \Psi \) at a point \( X_a \) which is its pole. Passing to polars, we see that \( X_a \) is \( \ell \cdot A \perp \). The line \( AX_a \) is tangent to \( \Psi \).

In a similar fashion, we obtain points \( X_b \) and \( X_c \) from the cones \( \psi_B \) and \( \psi_C \). Each of the points \( X_a, X_b, X_c \) lie on \( \ell \) and in \( \Psi \). Note that \( \ell \) lies in \( Q \perp \). Since \( \Psi \) does not contain lines, two of these three points must coincide, say \( X_b = X_c \). The lines \( BX_b \) and \( CX_c \) lie in \( Q \perp \) and are tangent to \( \Psi \) at \( X_b \) and \( X_c \), so they are tangent to the conic \( \Psi \cdot Q \perp \). Since these points of tangency coincide, the lines \( BX_b \) and \( CX_c \) are the same line \( BC \). But, \( BC \) cannot be tangent to \( \Psi \) by (S2). \( \text{QED} \)

6. Generalizations

The Apollonius contact problem in \( n \)-dimensional Euclidean space consists of finding those hyperspheres which are tangent to \( n + 1 \) given hyperspheres. For \( n = 3 \) this was recently studied by Fitz-Gerald [14], and was studied by the present authors in the context of the
geometry of an \( n \)-dimensional quadric \( \Psi \) in \( (n + 1) \)-dimensional projective space \( \mathbb{P}^{n+1} \) [13]. As in the two-dimensional case, hyperspheres and hyperplanes are represented by points in the exterior of \( \Psi \). Given \( n + 1 \) hyperspheres represented by \( A_1, A_2, \ldots, A_{n+1} \), the solutions to the Apollonius problem lie by pairs on \( 2n \) lines \( \ell^k \). More specifically, they are the intersections of these lines with a fixed tangent cone \( \psi_1 \) in \( \mathbb{P}^{n+1} \) with vertex \( A_1 \). As before, for a given line \( \ell^k \), we consider the conjugate \( (n - 2) \)-flat with respect to \( \psi_1 \) and show that its conjugate with respect to \( \Psi \) is a line that meets \( \Psi \) at the points which the solutions on \( \ell^k \) have in common with the hypersphere \( A_1 \).

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