ON THE GEOMETRY OF A QUADRATIC FORM: PENCILS AND THE APOLLONIUS CONTACT PROBLEM

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ABSTRACT: In this paper we consider the geometry of hyperspheres and hyperplanes in a non-degenerate metric affine space. We classify pencils of hyperspheres in terms of affine invariants obtained from a classification of lines in projective space which uses a suitable notion of square-classes for points of projective space. Using these ideas we treat the Apollonius contact problem in the context of the geometry of one non-singular quadric in projective space. From this we obtain an algorithm to determine the number of solutions to the Apollonius problem in this general setting. Key Words: hypersphere, hyperplane, Apollonius, quadratic, affine.

SOBRE LA GEOMETRÍA DE UNA FORMA CUADRÁTICA: HACES Y EL PROBLEMA DE CONTACTO DE APOLONIO

RESUMEN: En este trabajo se considera la geometría de hiperplanos e hipersuperficies en un espacio métrico afín no degenerado. Clasificamos los haces de hipersuperficies en términos de invariants algebraicos obtenidos de la clasificación de rectas en el espacio proyectivo usando la definición adecuada de clase cuadrática para puntos. Empleando estas ideas, consideramos el problema de contacto de Apolonio en el espacio proyectivo. Se obtiene un algoritmo para determinar el número de soluciones del problema de Apolonio en este contexto general. Palabras clave: hipersuperfície, hiperplano, Apolonio, cuadrática, afín.

0. INTRODUCTION

0.1. The study of circles in the Euclidean plane by means of invasive geometry is the origin of our subject. It is natural, and proves to be useful, to ask the classical questions in quite a general setting. So, we consider here the geometry of hyperspheres \( (x - a) \cdot (x - a) = \alpha \) and hyperplanes \( u \cdot (x - c) = 0 \) in a non-degenerate finite-dimensional metric affine space. Such an affine geometry is given by a non-degenerate quadratic form \( x \cdot x \) on a vector space \( V \) over a field \( K \) of characteristic not two, but otherwise arbitrary.

Early studies of the geometry of hyperspheres are described in the treatise by Coolidge [3]. Later, in the case that \( K \) is the field of complex numbers, hyperspheres were studied by Scherk [14], and, in the case that \( K \) is real numbers and \( V \) is \( n \)-dimensional Euclidean space, by Alexander [1]. Recently, geometry and physics has made use of the invasive geometry of hyperspheres. When \( V \) is a Minkowski space, the hyperspheres are given by an indefinite metric. For example, if \( n = 3 \) and the signature of the bilinear form is \( -+ \), then the hyperspheres are one-sheeted hyperboloids, cones and planes in that space. For \( n = 4 \) and signature \( -++ \), this space is widely used in physics. See, for example, Fillmore [5]. The general case has been considered by Lester [11].

We focus here on a general notion of tangency and its connections with pencils of hyperspheres and square-classes. Hyperspheres and hyperplanes will be represented by points in a projective space and we use this technique to explore the relationship between tangency and square-classes of \( K \) which are associated to hyperspheres. We were led to these considerations by studying a generalization of the classical Apollonius contact problem.

0.2. The classical Apollonius contact problem consists of finding the circles which are tangent to three given circles. It has been studied by many people from many viewpoints, one of which is the classification of the solutions. Muirhead [12] carries out the first classification of the solutions and Bruen et al. [2] use invasive geometry to obtain another method of classification.


Hohenberg solves the Apollonius problem in Euclidean \( n \)-dimensional space with additional assumptions on the independence of the centers of the \( n + 1 \) given hyperspheres. We should remark that his notion of tangency is an oriented tangency. His solution is set in the context of Euclidean geometry and focuses on first finding the radii of the solutions. Hohenberg cleverly obtains beautiful formulas for these radii using some old expressions, to be found in the treatise of Schoute [15], for volumes related to certain sim-
plices. The simplices which he uses depend on the relative orientations of the given hyperspheres and consequently the computations involved to determine the number of solutions for a given configuration are necessarily long.

Iwata and Naitō develop some formulas in the same style as those of Hohemberg, again using geometry of simplices in Euclidean n-space. These authors also treat other problems concerning hyperspheres in several articles which may be located through the extensive bibliography in Wilker [17].

The problem of Apollonius does not seem to have been considered systematically for pseudo–Euclidean spaces or for fields other than the reals. This is not as difficult as might be thought. The key is to avoid the extractions of the square roots used to assign a “radius” to a hypersphere. One should instead work in terms of a “square–radius” which need not be the square of an element of the field, and consider hyperspheres to be “of the same square–class” when the ratio of their “square–radius” is a square.

The generalized Apollonius contact problem consists of finding the hyperspheres which are tangent to \( n + 1 \) given hyperspheres in an \( n \)-dimensional pseudo–Euclidean space. It is here treated from the viewpoint of projective geometry.

Projective geometry is the most natural setting for a problem which is invariant under inversions in hyperspheres. By treating the Apollonius problem from this viewpoint, we obtain a computational algorithm which easily yields the number of solutions for a given configuration in a pseudo–Euclidean space over any field.

Section 3 is a compendium of standard facts needed later about projective geometry of a quadric, but with special emphasis on the connections with the square–classes of the field \( K \).

In Section 3, we use stereographic projection to establish the correspondence between hyperspheres and hyperplanes of affine geometry as loci of certain quadratic and linear equations which, in order to be non–empty, are attatched to certain square–classes of \( K \). In establishing a working definition of tangency, we are led to consider pencils of hyperspheres and present there several equivalent notions of tangency, some geometric and some algebraic.

Section 2 is a compendium of standard facts needed later about projective geometry of a quadric, but with special emphasis on the connections with the square–classes of the field \( K \).

In Section 4, we consider the generalized Apollonius contact problem of finding the spheres (i.e., hyperspheres and hyperplanes) tangent to \( n + 1 \) given spheres in \( V \). This is done in the projective setting by studying the intersection of cones, which, even in this general setting, is reduced to the intersection of certain line with one cone or to the question whether certain points lie on the quadric.

Section 5 contains an algorithm that counts the number of solutions of the Apollonius problem for almost any configuration of spheres in \( V \).

Throughout, we give examples of the most unexpected and striking differences between the geometry arising from a general quadratic form, and the familiar geometry of circles in the Euclidean plane or spheres in space.

0.4 For the notation for this paper we will let \( V \) be a vector space of dimension \( n \) over a field \( K \) of characteristic not two, and denote by \( x \cdot y \) the polarization of a non–degenerate quadratic form on \( V \). Other than these assumptions, the field and quadratic form may be arbitrary.

A hypersphere is a non–empty locus in \( V \) of an equation of the form \( (x - a) \cdot (x - a) = \alpha \), where \( a \) is in \( V \) and \( \alpha \) is in \( K \). By identifying the affine space with the vector space, we write \( x \cdot x = \alpha \) for \( (x - 0) \cdot (x - 0) = \alpha \). Note that we will deal carefully with the problem that equations such as these may have empty loci. Similarly, a hyperplane is the (always) non–empty locus of a linear equation. We will later attach square–classes to non–singular hyperspheres and hyperplanes.

Geometry will be done in three spaces: the space \( V \) of dimension \( n \) where we will use the terms hypersphere, hyperplane, ..., as well as in a metric affine space \( V^+ \) of dimension \( n + 1 \) where we use affine quadric, n–plane, \( (n - 1)–\)plane, k–plane, ..., and in the projective space \( P^n \) containing \( V^+ \) where we use quadric, prime, secon
dam, ..., k–flat. ... The term sphere will be reserved for certain points of \( P^n \) which will represent hyperspheres and hyperplanes of \( V \).

1. HYPERSPHERES AND HYPERPLANES

1.1 Let \( K \) be a field of characteristic not two. The square–class of a non–zero element \( \alpha \) in \( K^* \) is \( \alpha \) modulo \( (K^*)^2 \) in \( K^*/(K^*)^2 \), so two non–zero elements of \( K \) are in the same square–class if their quotient is a square in \( K \).

Let \( V \) be an \( n \)-dimensional vector space, \( n \geq 2 \), equipped with a non–singular quadratic form \( x \cdot x \) with polarization \( x \cdot y \). An element \( \alpha \) in \( K \) will be called \( \textit{representable} \) (by the quadratic form in \( V \)) if \( \alpha = z \cdot z \) for some \( z \) in \( V \). If \( \alpha \neq 0 \), clearly this property depends only on the square–class of \( \alpha \).

In the case that the field \( K \) is the real numbers \( \mathbb{R} \), there are exactly two square–classes. Positive definite quadratic forms represent precisely real numbers with square–class of +1. If the quadratic form is indefinite then every real number is representable. These same assertions hold for the field generated over the rationals by ruler and compass constructions, especially for the classical case \( x_1^2 + x_2^2 + \ldots + x_n^2 \). In the case the field \( K \) is the complex numbers \( \mathbb{C} \), there is only one square–class and all complex numbers are representable by any form.

In the case that the field \( K \) is the rational numbers \( \mathbb{Q} \), every square–class contains a unique, positive or negative, square–free integer. The distinct prime factors of this integer, together with its sign, specify the square–class. For the most common quadratic forms from classical Euclidean and Minkowski geometry, that is, the forms \( \pm x_1^2 \pm \ldots \pm x_n^2 \),
The Apollonius contact problem

\( n \geq 2 \), we may immediately infer the representability of an element of \( Q \) using the following facts from arithmetic:

- Every rational number \( \alpha \) is the difference of the two squares \( \left( \frac{\alpha + 1}{2} \right)^2 - \left( \frac{\alpha - 1}{2} \right)^2 \);
- The rational number \( \alpha \) is the sum of two rational squares if and only if \( \alpha \) has the same square-class as a positive square–free integer whose prime factors are 2 or are congruent to 1 modulo 4. This is an elementary consequence of the condition, due to Fermat, for an integer to be the sum of two squares;
- The rational number \( \alpha \) is the sum of three rational squares if and only if \( \alpha \geq 0 \) and \( \alpha \) does not have the same square-class as \( 4^a(8b - 1) \) with integers \( a \geq 0 \) and \( b \geq 1 \). This is a theorem due to Gauss [16, p. 45, Lemma A];
- Every positive rational number is the sum of four rational squares. This is a theorem due to Lagrange [16, p. 47, Corollary 1].

In the case that the field \( K \) is the finite field \( \mathbb{F}_q \) with \( q \) elements, where \( q \) is the power of an odd prime, then \( \mathbb{F}_q \) has two square-classes (although not always given by 1 and \(-1\)) [16, p. 6]. Every element in \( \mathbb{F}_q \) is the sum of two squares; this can be proven by counting. Every element of \( \mathbb{F}_q \) is the difference of two squares as in the rational case.

One can make similar statements for other fields and non–degenerate quadratic forms [16]. The fields and quadratic forms above will suffice to provide the examples for the geometry that follows.

Finally, we note that the above facts from arithmetic also yield conditions for the existence of (non–zero) isotropic vectors for forms \( \pm x_1^2 \pm x_2^2 \pm \ldots \pm x_n^2 = 0 \) by examining \( \pm (x_1/x_2)^2 \pm \ldots \pm (x_{n-1}/x_n)^2 = 1 \).

We must keep in mind that it is the quadratic form on \( V \) which distinguishes between representable and non–representable square-classes in \( K \). In the special case that \( V \) contains non–zero isotropic vectors, every square–class is representable.

1.2 Given a in the affine space \( V \) and \( \alpha \) in \( K \), a non–empty set of points \( x \) in \( V \) which satisfy the equation \( (x - a) \cdot (x - a) = \alpha \) will be called the hypersphere with center \( a \) and square-radius \( \alpha \). (N.B.: \( \alpha \) need not be a square in \( K \)). In Section 3.7 we will touch briefly on the notion of “empty” or “imaginary” hyperspheres and the corresponding invariance, but otherwise hyperspheres are to be non–empty. A hypersphere is called non–singular or singular as its square-radius \( \alpha \) is \( \neq 0 \) or \( = 0 \). Let us now highlight some properties of hyperspheres for some general \( V \).

If the quadratic form on \( V \) has (non–zero) isotropic vectors, then every equation \( (x - a) \cdot (x - a) = \alpha \) determines a hypersphere. If the quadratic form is anisotropic, there will be equations with empty loci, and in this case the center of a singular hypersphere is its only point.

More interestingly, if the quadratic form has (non–zero) isotropic vectors, then a singular hypersphere contains \( n + 1 \) affinely independent points of \( V \). This is a consequence of the following observation about quadratic forms:

A non–degenerate quadratic form having a (non–zero) isotropic vector will have \( n \) isotropic vectors which form a basis of \( V \).

For the proof, recall that the characteristic of \( K \) is not two. Since there is one isotropic vector there is a hyperbolic plane with basis \( \{ v_1, v_2 \} \) having \( v_1 \cdot v_1 = 1 \), \( v_1 \cdot v_2 = 0 \) and \( v_2 \cdot v_2 = -1 \). Let \( \{ v_3, \ldots, v_n \} \) be an orthogonal basis of \( \langle v_1, v_2 \rangle \). The vectors

\[
\begin{align*}
\langle v_1 \rangle : v_1 + v_2, & \quad v_1 - v_2 \quad \text{and} \quad v_1 \frac{1 - v_1 \cdot v_1}{2} + v_2 \frac{1 + v_1 \cdot v_1}{2} + v_i,
\end{align*}
\]

for \( i = 3, 4, \ldots, n \), are then \( n \) isotropic independent vectors.

Irrespective of \( V \) having isotropic vectors or not, we have:

**Theorem 1.1.** A non–singular hypersphere contains \( n + 1 \) affinely independent points, except when \( n = 2 \) and \( K \) is the field \( F_3 \) of three elements.

**Proof:** This proof is based on what will be later recognized as “stereographic projection”. Without loss of generality, by translating to the origin, consider the non–singular hypersphere centered at the origin and defined by \( x \cdot x = \alpha \), and let \( e \) be a point on it. The map

\[
\langle e \rangle \setminus \{ x \cdot x = \alpha \} \rightarrow \{ x \cdot x = \alpha \} \setminus \{ e \cdot x = \alpha \}
\]

given by

\[
\begin{align*}
\nu & \sim \nu \frac{2\alpha}{\nu \cdot \nu + \alpha} + e \frac{\nu \cdot e - \alpha}{\nu \cdot \nu + \alpha},
\end{align*}
\]

for \( \nu \) in \( \langle e \rangle \), is a bijection with inverse \( \nu + e \nu \sim \nu (1 - \nu) \) for \( \nu \) in \( \langle e \rangle \) and \( \nu \) in \( K \). The points \( v, \nu \) and \( v \frac{2\alpha}{\nu \cdot \nu + \alpha} + e \frac{\nu \cdot e - \alpha}{\nu \cdot \nu + \alpha} \) are collinear. Let \( v_1, \ldots, v_n \) be independent vectors of \( \langle e \rangle \). For those \( i \) for which \( v_i \cdot e = -\alpha \), replace \( e \) by \( e \lambda \) where \( \lambda^2 \neq 1 \) (such \( \lambda \) in \( K \) exists since the field is not \( F_3 \)); now \( v_i \cdot e = \lambda \) is not zero for all \( i \). The \( n + 1 \) points

\[
\begin{align*}
\pm e \quad \text{and} \quad v_i \frac{2\alpha}{v_i \cdot v_i + \alpha} + e \frac{v_i \cdot e - \alpha}{v_i \cdot v_i + \alpha},
\end{align*}
\]

for \( i = 1, 2, \ldots, n - 1 \), which are \( e \) and the images of zero and \( v_1, \ldots, v_{n-1} \) under the bijective map, are the independent points of the hypersphere given by \( x \cdot x = \alpha \). This is the case since the \( n \) vectors from \( e \) to the other points are

\[
\begin{align*}
-2e \quad \text{and} \quad v_i \frac{2\alpha}{v_i \cdot v_i + \alpha} + e \frac{-2\alpha}{v_i \cdot v_i + \alpha},
\end{align*}
\]

for \( i = 1, 2, \ldots, n - 1 \), which are clearly linearly independent in \( V \). This establishes our statement in the case that \( K \) is not \( F_3 \).

Suppose now that \( K \) is \( F_3 \). In the case that \( n = 3 \), by appropriate choice of basis, the non–singular quadratic form can be written as \( x_1^2 + x_2^2 + x_3^2 \) or \( -x_1^2 + x_2^2 + x_3^2 \) and it is easy to check (by computing points) that in either case
every hypersphere has four independent points. For \( n > 3 \), consider a non-singular quadratic form

\[
\delta x_1^2 + x_2^2 + x_3^2 + \delta_4 x_4^2 + \ldots + \delta_n x_n^2
\]

where \( \delta \) and the \( \delta_i \) are 1 or \(-1\). For a hypersphere centered at the origin and square-radius \( \alpha \), the following \( n \) vectors in \( V \) are independent and go from the origin to points of the hypersphere:

\[
v_i = (\xi_i, \eta_i, \zeta_i, 0, \ldots, 0), \quad i = 1, 2, 3,
\]

where the \((\xi_i, \eta_i, \zeta_i)\) are three independent vectors so lying on the hypersphere of equation \( \delta_1 x_1^2 + \delta_2 x_2^2 + \delta_3 x_3^2 = \alpha \), and

\[
v_i = (\xi, \eta, \zeta, 0, \ldots, 0, 1, 0, \ldots, 0), \quad i = 4, \ldots, n
\]

where \( 1 \) is in the \( i \)-th position and \((\xi, \eta, \zeta)\) is any one solution of \( \delta_1 x_1^2 + \delta_2 x_2^2 + \delta_3 x_3^2 = -\delta_1 + \alpha \). The \( n = 1 \) points \( 2v_1, v_1, \ldots, v_n \) then lie on the hypersphere and are independent. This finishes the proof of our assertion.

If \( n = 2 \) and the field has three elements, it is easy to produce non-singular hyperspheres which contain only two points.

Although throughout we almost exclusively work with the equations of hyperspheres, it is indeed the case that the points of a non-singular or singular hypersphere determine this equation. For, consider a hypersphere of equation \((x - a) \cdot (x - a) = \alpha\). If \((x - b) \cdot (x - b) = \beta\) is a second equation with the same locus, then this locus is contained in the common of \(2(b - a)\cdot x = (b \cdot b - \beta) - (a \cdot a - \alpha)\). If the hyperspheres contain \( n + 1 \) independent points, then the intersection of the two hyperspheres is a sphere consisting of its center only, or necessarily \( n = 2 \) and the field \( K \) consists of three elements. In the former case, if \((x - a) \cdot (x - a) = 0\) is uniquely determined by the scalar product \(a\), since the quadratic form \(x \cdot x\) has no isotropic vectors. If the latter case, \( V \) has nine points and a hypersphere may have as few as two points, but nevertheless a direct calculation of the possibilities shows that a hypersphere determines its equation.

1.3 We define the square-class of a non-singular hypersphere to be the square-class of its square-radius.

A hyperplane (or hypersphere of "infinite radius") is the set of points \(x\) in \(V\) which satisfy an equation \(u \cdot (x - c) = 0\), where \(c\) is a point of \(V\) and \(u\) is a vector. It is non-singular or singular as \( u \cdot u \neq 0 \) or \( u \cdot u = 0 \). Of course \( u \cdot u \) need not be a square in \(K\). The hyperspheres \((x - c) \cdot (x - c) = u \cdot u t^2\) classically have as limit the hyperplane \(u \cdot (x - c) = 0\), when \( t \) becomes infinite. The square-class of a non-singular hyperplane \(u \cdot (x - c)\) is defined to be that of the scalar \(u \cdot u\) in agreement with the common square-class of the hyperspheres of such a family.

The square-class of non-singular hyperplanes and non-singular hyperspheres are representable by the quadratic form on \(V\). The tangent hyperplane to the hypersphere \((x - a) \cdot (x - a) = \alpha\) at its point \(p\) is defined by \((p - a) \cdot (x - a) = \alpha\). Note that the tangent hyperplane exists at every point of a hypersphere except at the centers of singular hyperspheres. The line joining \(p\) to a point in the tangent hyperplane to the given hypersphere either meets it in exactly one point or lies entirely in the hypersphere.

Example:

(a) In three-dimensional Minkowski space, the hyperplane \(x_1^2 + x_2^2 - x_3^2 = 1\) is met by its tangent hyperplane \(x_1 = 1\) at \((1, 0, 0)\) in the lines where \(x_2 = 1\) and \(x_3^2 - x_3^2 = 0\).

1.4 In the spirit of Salmon [13], we say that two distinct hyperspheres \((x - a) \cdot (x - a) = \alpha\) and \((x - b) \cdot (x - b) = \beta\), possibly singular, determine a pencil of hyperspheres whose members have as equations linear combinations of those of the given hyperspheres:

\[
((x - a) \cdot (x - a) - \alpha)(1 - t) + ((x - b) \cdot (x - b) - \beta)t = 0
\]

Here \(t\) lies in \(K\), but one includes in the pencil the radical hyperplane given by the linear equation

\[
-(x - a) \cdot (x - a) - (x - b) \cdot (x - b) = 0,
\]

as corresponding to "\(t = \infty\)". Note that the above family of equations may contain members with empty loci. Such pencils may be in fact defined via powers of points with respect to hyperspheres, as carried out, for instance, for the classical case in Court [4].

By completing squares, this family of equations becomes

\[
(x-a(1-t)-bt)(x-a(1-t)-bt) - \alpha(1-t) + \beta t - \delta t(1-t) = 0
\]

where \(\delta = (a-b)(a-b)\), from which we recognize equations for hyperspheres with centers \((1-t)+bt\), on the line of centers of the given hyperspheres, and having square-радиус

\[
(1-t) + \beta t - \delta t(1-t).
\]

The pencil of hyperspheres determined by the distinct hyperspheres \((x-a) \cdot (x-a) = \alpha\) and \((x-b) \cdot (x-b) = \beta\) contains \(0, 1, 2, 0\) or \(\infty\) singular hyperspheres which correspond to the roots in \(K\) of equation (1.1), whose discriminant we denote by

\[
\Delta = (-\delta + \beta - \alpha)^2 - 4\delta \alpha = (\delta - \alpha - \beta)^2 - 4\alpha \beta.
\]

In case \(\delta \neq 0\), (1.1) is a quadratic equation having \(2, 1\) or 0 roots in \(K\) according as \(\Delta\) is a non-zero square, \(\Delta\) equals zero, or \(\Delta\) is a (non-zero) non-square. This corresponds to two distinct singular hyperspheres in the pencil, exactly one such, or none, respectively.

In case \(\delta = 0\), (1.1) has \(\infty, 1, 0\) solutions according as \(\alpha = \beta = 0\), \(\alpha \neq \beta\), or \(\alpha = \beta \neq 0\).

Example:

(b) The hyperspheres \(x_1^2 - x_2^2 = 0\) and \((x_1 - 1)^2 - (x_2 - 1)^2 = 0\) in the real Minkowski plane determine a pencil having every hypersphere singular, while \(x_1^2 - x_2^2 = 3\) and \((x_1 - 1)^2 - (x_2 - 1)^2 = 3\) determine a pencil having no singular hyperspheres.
1.5 Of various equivalent definitions of tangency, one of them is especially suitable for dealing with hyperspheres in this general setting. Classically, two hyperspheres \((x - a) \cdot (x - a) = \rho^2 \) and \((x - b) \cdot (x - b) = \sigma^2 \) are tangent if the square of the distance between their centers is the square of the difference or sum of their radii: \((a - b) \cdot (a - b) = (\rho^2 + \sigma^2)\), or, equivalently,

\[(a - b) \cdot (a - b) - \rho^2 - \sigma^2 = 4\rho^2\sigma^2.\]

In general, we define the hyperspheres \((x - a) \cdot (x - a) = \alpha\) and \((x - b) \cdot (x - b) = \beta\) to be tangent if

\[\((a - b) \cdot (a - b) - \alpha - \beta = 4\alpha \beta.\]  \hspace{1cm} (1.2)

Since (1.2) is equivalent to \(-((a - b) \cdot (a - b) + \alpha - \beta) = 4(a - b) \cdot (a - b)\) and similarly with \(4(a - b) \cdot (a - b)\beta\), (1.2) gives that the three elements

\[(a - b) \cdot (a - b), \quad \alpha \quad \text{and} \quad \beta\]

of \(K\) satisfy exactly one of the following:

- all three are non-zero and lie in the same representable square-class;
- one is zero and the other two are equal and not zero;
- all three are zero.

In particular, if two hyperspheres are tangent, either both belong to the same representable square-class or one or both are singular.

Since the above is going to be our working definition of tangency, we investigate its geometric consequences.

Let \((x - a) \cdot (x - a) = \alpha\) and \((x - b) \cdot (x - b) = \beta\) be two hyperspheres such that \(\delta = (a - b) \cdot (a - b)\) is not zero. The radical hyperplane of such a non-singular and the line of centers joining \(a\) to \(b\) meets it at the unique point

\[m = a + \delta + \beta = \frac{\delta + \alpha - \beta}{2\delta},\]

which we call the radical trace.

For the proof that follows, we note that \(m - a = \frac{\delta + \alpha - \beta}{2\delta}\), so

\[(m - a) \cdot (m - a) = \frac{4\delta^2}{(\delta + \alpha - \beta)^2} - \frac{4\alpha \beta}{\delta} = \frac{4\delta^2}{(\delta + \alpha - \beta)^2} - 4\alpha \beta + \alpha - \beta = \frac{4\delta^2}{(\delta + \alpha - \beta)^2} - 4\alpha \beta.\]

(1.3)

and, by symmetry,

\[(m - b) \cdot (m - b) - \beta = \frac{(\delta - \alpha + \beta)^2 - 4\alpha \beta}{2\delta} = \frac{4\delta^2}{(\delta - \alpha + \beta)^2} - 4\alpha \beta.\]

(1.4)

for every \(x\) in \(V\).

\textbf{Theorem 1.2.} Given two hyperspheres with equations \((x - a) \cdot (x - a) = \alpha\) and \((x - b) \cdot (x - b) = \beta\) such that \(\delta = (a - b) \cdot (a - b)\) is not zero, the following are equivalent:

(i) There exists a point \(p\) lying on both hyperspheres at which the tangent hyperplanes \((p - a) \cdot (x - a) = \alpha\) and \((p - b) \cdot (x - b) = \beta\) coincide, or one of the hyperspheres is singular and its center lies on the other hypersphere.

(ii) The radical trace point \(m\) lies on one of the hyperspheres (and consequently on both).

(iii) The pencil determined by the two hyperspheres contains a unique singular hypersphere.

(iv) The two hyperspheres are tangent, that is,

\[(\delta - \alpha - \beta)^2 = 4\alpha \beta.\]

\textbf{Proof.} (ii), (iii) and (iv) are equivalent by (1.3) and equation (1.1).

Now assume (i). Suppose first that the two hyperspheres share their tangent hyperplane at \(p\). This means that there exists \(\lambda \neq 0\) such that

\[(p - b) \cdot (x - b) - \beta = ((p - a) \cdot (x - a) - \alpha)\lambda\]

for all \(x\) in \(V\). Comparing the coefficients of \(x\) gives \(p - b = (p - a)\lambda\). Since \(\lambda \neq 0\), and \(p = \frac{\lambda - 1}{\lambda - 1} b\), so it follows that \(p\) lies on the line of centers. Since \(p\) also lies on both hyperspheres, it lies on the radical hyperplane. Therefore (ii) follows because \(p\) coincides with the radical trace point \(m\). If the second hypersphere is singular and its center lies on the first, we have \((b - a) \cdot (b - a) = \alpha\). Then \(\delta = \alpha\) and hence \(m = b\), which again implies (ii).

Now assume (iv). It cannot happen that both hyperspheres are singular, for otherwise \(\alpha - \beta = 0\) and (iv) would imply \(\delta = 0\). Suppose that one of the hyperspheres is singular, say \(\beta = 0\). Then by (iv), \(\delta = \alpha\) or \((b - a) \cdot (b - a) = \alpha\) and the singular hypersphere has its center on the other. Finally, suppose that both hyperspheres are non-singular. By (ii), \(m\) lies on both hyperspheres. Now, neither coefficient \(\frac{\delta - \alpha + \beta}{2\delta}\) nor \(\frac{\delta + \alpha - \beta}{2\delta}\) can be zero. For if, say, \(\frac{\delta - \alpha + \beta}{2\delta} = 0\) so \(\delta + \alpha - \beta = 1\), then \(m = b\) and hence \(\beta\) would be 0, because the right hand side of identity (1.4) is zero. Now it is evident from (1.4) that both hyperspheres have the same tangent hyperplane at \(m\). \(\blacksquare\)

\textbf{1.6 Concerning tangency:}

\textbf{Remark 1.} The condition (1.2) for tangency of two hyperspheres, that is, the vanishing of \((\delta - \alpha \beta)^2 - 4\alpha \beta\) is a polynomial in three variables with integer coefficients and hence is unchanged when the scalars and the space \(V\) with its quadratic form are extended to any field containing \(K\). In case \(\delta \neq 0\), two tangent hyperspheres, when viewed with their additional points in the extension of \(V\), must have their point of tangency in \(V\) itself. For, their point of tangency is the radical trace.
Remark 2. In the Euclidean case, tangency between a singular and a non-singular hypersphere amounts merely to a point lying on a hypersphere. This is analogous in other signatures as the following example shows.

Example:

(c) In the real Minkowski plane consider the pencil given by setting

\[(x_1^2 - x_2^2 - 1)(1-t) + ((x_1 - 1)^2 - x_2^2) t = 0\].

One may use condition (iv) of the Theorem 1.2 to verify the tangency of any two hyperspheres of this pencil:

\[(s-t)^2 - (1-s)^2 - (1-t)^2)^2 = 4(1-s)^2(1-t)^2\].

The unique singular hypersphere of the pencil is \[(x_1 - 1)^2 - x_2^2 = 0\] and its center \((1, 0)\) lies on each of the hyperspheres.

Remark 3. In the case that \(\delta = (b - a) \cdot (b - a)\) is zero, the tangency of two hyperspheres does not imply the existence of a tangent point. In fact, given any two non-concentric hyperspheres \((x - a) \cdot (x - a) = \alpha \) and \((x - b) \cdot (x - b) = \beta\) for which \(\delta = 0\), the line of centers is parallel to the radical hyperplane. (That is, a normal to the hyperplane and the direction vector of the line are orthogonal with respect to the bilinear form \(x \cdot y\) of the vector space \(V_{\psi}\).) Two such hyperspheres are tangent if and only if \(\alpha = \beta\), and this is equivalent to the line of centers lying on the radical hyperplane. Further, in the case that \(\delta = 0\), two non-concentric tangent hyperspheres have one or more points in common if and only if their common square-radius \(\alpha\) is representable by a vector orthogonal to the vector \(b - a\) between their centers.

Example:

(d) In the real Minkowski space, the hyperspheres

\[x_1^2 + x_2^2 - x_3^2 = \alpha\]

and

\[(x_1 - 1)^2 + x_2^2 - (x_3 - 1)^2 = \alpha\]

are tangent, and have in common the points \((t, x_2, t)\), where \(x_2^2 = \alpha\).

2. A QUADRIC IN PROJECTIVE SPACE

2.1 Let \(W\) be an \((n+2)\)-dimensional vector space over a field \(\mathbb{K}\) of characteristic not two and \(\mathbb{P}^{n+1} = \mathbb{P}(W)\) the corresponding projective space whose points are one-dimensional subspaces \((P)\) spanned by vectors \(P\) of \(W\). Let \(\Psi\) be a non-degenerate quadric in \(\mathbb{P}^{n+1}\). This quadric is given by a non-degenerate quadratic form on \(W\) which is determined only up to a non-zero "scale" factor.

Denote by \(O(W)\) the group of linear isomorphisms preserving one, say \((X|X)\), (and hence all) of these quadratic forms. By Witt's theorem [16, p. 31], the corresponding group \(PO(W)\) of projective transformations is transitive on \(\Psi\).

Given two points in \(\mathbb{P}^{n+1}\), say \((A)\) and \((B)\), not lying on \(\Psi\), we say that they belong to the same square-class if \((B|B) = \lambda^2 (A|A)\) for some non-zero \(\lambda\) in \(\mathbb{K}\). Clearly this is a notion independent of the scaling of the quadratic form. As yet we do not assign a square-class to a point of \(\mathbb{P}^{n+1}\), but this will be done in the next section.

The following proof, although trivial, focuses on things to be used later.

Proposition 2.1. Two points \((A)\) and \((B)\), not lying on \(\Psi\), are in the same square-class if and only if they lie on the same orbit \(PO(W)\) acting on \(\mathbb{P}^{n+1}\).

Proof: Suppose \((A)\) and \((B)\) lie in the same square-class. Let \(\lambda\) in \(\mathbb{K}\) satisfy \((B|B)/(A|A) = \lambda^2\). Then the vectors \(K = B - A\lambda\) and \(K' = B + A\lambda\) may not both be isotropic. If \((K|K)\) is not zero, define \(\sigma_K\) by \(\sigma_K(X) = X - \frac{2(K|X)}{(K|K)} (K|X)\). A computation shows that \(\sigma_K(A) = B\lambda^{-1}\) and that \((\sigma_K(X)|\sigma_K(X)) = (X|X)\), as well as \(\sigma_K(\sigma_K(X)) = X\) for all \(X \in W\). The converse is trivial.

See Lam [10, Chapter 1] for a full discussion of \(O(W)\) and a non-projective version of the above proposition.

There are four kinds of intersections that lines in \(\mathbb{P}^{n+1}\) can have with the quadric \(\Psi\). The line joining two distinct points \((A)\) and \((B)\) intersects \(\Psi\) in two points, no points, or is tangent according as \(-(A|B)(B|B) + (A|B)^2\) is a non-zero square, a non-square, or zero. In the last case the line meets the quadric in a unique point or lies entirely within it, according as the restriction of the quadratic form \((X|X)\) to the two dimensional subspace \((A, B)\) has rank one or zero.

2.2 Two points \((A)\) and \((B)\) are called \textit{conjugate} with respect to \(\Psi\) if \((A|B) = 0\). The n-flat which is the locus of points \((X)\) which are conjugate to the points \((P)\) is given by the homogeneous equation \((P|X) = 0\) and is called the polar \(\mathbb{P}^1\) of the point \((P)\). Since the quadratic form is non-degenerate, every prime is obtained in this manner from a point called its pole.

A prime is \textit{tangent} if its pole \((P)\) lies on \(\Psi\). The point \((P)\) is referred to as the point of tangency. Any line tangent to \(\Psi\) at \((P)\) necessarily lies in \(\mathbb{P}^1\).

For \((P)\) in \(\mathbb{P}^{n+1}\), it will be later useful to distinguish \(\Psi \cap \mathbb{P}^1\) being empty or not empty. The latter case occurs precisely when the restriction of the quadratic form \((X|X)\) to \(\mathbb{P}^1\) has isotropic vectors. Of course, this restriction is degenerate when \((P)\) lies on \(\Psi\).

Given an arbitrary point \((P)\) in \(\mathbb{P}^{n+1}\), the cone \(\psi_P\) with vertex \((P)\) is the set of points \((X)\) such that \((X|P)^2 - (X|X)(P|P) = 0\). If \((P)\) is on \(\Psi\), then \(\psi_P\) coincides as set with the tangent plane \((P)^*\). In the case that \((P)\) is not on \(\Psi\), the points of \(\psi_P\), excluding those in \(\Psi\), belong to the same square-class as that of the vertex \((P)\). Hence, if \((A)\) and \((B)\) lie in distinct square-classes, then \(\psi_A \cap \psi_B \subset \Psi\).

As a matter of fact, in the classical case this intersection is
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empty, but in the general setting it may be not empty, as an example later will show.

For any \( \langle P \rangle \) in \( \mathbb{P}^{n+1} \), one has

\[
\Psi \cap \langle P \rangle^\perp = \Psi \cap \psi_P.
\]  

(2.1)

As a consequence, \( \Psi \cap \langle P \rangle^\perp \) is non-empty if and only if \( \psi_P \) contains points other than its vertex \( \langle P \rangle \).

Finally we remark that, if \( \langle P \rangle \) lies in \( \mathbb{P}^{n+1} \setminus \Psi \), then \( \Psi \cap \langle P \rangle^\perp = \Psi \cap \psi_P = \langle P \rangle^\perp \cap \psi_P \).

2.3 The action of \( PO(W) \) on \( \mathbb{P}^{n+1} \) sends lines to lines. Orbits are determined by the behavior of the restriction of the quadratic form \((X|X)\) to two-dimensional subspaces of \( W \).

Any two lines lying wholly in \( \Psi \) belong to the same orbit by Witt's theorem.

Consider now lines tangent to but not lying in \( \Psi \). Two points, not in \( \Psi \), of such a line are in the same square-class (since \(- (A|A)| (B|B) | + | (A|B) = 0 \)). By Witt’s theorem, two such lines are in the same orbit exactly when a point not on the quadric, of one line is of the square-class as some point of the other line.

Lines which meet \( \Psi \) in exactly two points again, by Witt's theorem (for hyperbolic two-spaces), lie in the same orbit.

In the remaining case of lines not meeting the quadric, there may be several orbits. By Witt’s theorem, two lines lie in the same orbit if and only if the restriction of the quadratic form to the corresponding two-dimensional subspaces are isometric. This is an equivalence problem for non-degenerate binary quadratic forms. It is well known [10, p. 20] that two non-degenerate binary quadratic forms are equivalent if and only if their Gram determinants belong to the same square-class and the forms represent a common non-zero element in the field \( \mathbb{K} \).

3. REPRESENTATIONAL GEOMETRY

3.1 In the classical case of the real Euclidean \( n \)-space \( V \), stereographic projection, from the vector space \( V \) of dimension \( n \) to an affine quadric in the metric affine space \( V^+ \) of dimension \( n+1 \) containing \( V \), is well known. It carries hyperspheres and hyperplanes of \( V \) to the intersection of the affine quadric with \( n \)-planes of \( V^+ \). The poles of these \( n \)-planes are points of \( V^+ \) which represent hyperspheres and hyperplanes of \( V \) [9, p. 195]. In order to represent all hyperspheres and hyperplanes of \( V \), the affine space \( V^+ \) and the affine quadric are made part of a projective space and a (projective) quadric, and the stereographic projection is carried out there. In the setting of general \( V \), we carry out the affine description only far enough to provide appropriate interpretations, but then immediately pass to projective space, where the descriptions are simpler and complete.

3.2 Given the \( n \)-dimensional space \( V \), with non-degenerate quadratic form \( x \cdot x \), we define \( V^+ \) to be the \( n+1 \)-dimensional direct sum consisting of vectors \( x' = x + cy \) and \( y \) in \( K \). The quadratic form on \( V^+ \), \( x' \cdot x' = x \cdot x + cy^2 \), where \( c \) in \( \mathbb{K} \) is not zero and need not be a square, is also non-degenerate and makes the direct sum orthogonal. The affine quadric \( \Psi_{aff} \) given by \( x \cdot x + cy^2 = \rho^2 \), where \( \rho \) is not zero, is the analog of the “sphere” for classical stereographic projection with \( \epsilon \rho \) as the “north pole” and \( V \) as the “equatorial plane”. Here \( \epsilon \cdot \epsilon = \epsilon \) plays the role of the “unit square-radius” and \( \rho \) that of the “radius” of \( \Psi_{aff} \). The parameters \( \epsilon \) and \( \rho \) are not set equal to 1 so as to be able to keep track of square-classes and the degrees of homogeneous elements.

The notions of pole and polar obtain in this affine setting. The polar \( n \)-plane (with respect to \( \Psi_{aff} \)) of \( x_0' \) in \( V^+ \) is \( x_0' \cdot x' = \epsilon \rho^2 \); \( x_0' \) is the pole. Any \( n \)-plane not through the origin is of this form. An \( n \)-plane is tangent to \( \Psi_{aff} \) exactly when its pole lies on \( \Psi_{aff} \).

For a point \( p \) in \( V \), the second intersection of the line joining it and \( \rho \) with \( \Psi_{aff} \) is the point

\[
\frac{x_0' \cdot x' + \epsilon \rho^2}{x_0' \cdot x' + \epsilon \rho^2} = \frac{x_0' \cdot x + \epsilon \rho^2}{x_0' \cdot x + \epsilon \rho^2}.
\]

This defines a map from \( V \setminus \{ x \cdot x + \epsilon \rho^2 = 0 \} \) to \( \Psi_{aff} \setminus \{ y = \rho \} \) called stereographic projection. Its inverse sends \( x + \epsilon y \) in \( \Psi_{aff} \setminus \{ y = \rho \} \) to the point \( x_0' \)

\[
\frac{p}{\rho - y}.
\]

Example:

(e) The appropriate sketch, when \( V \) is the real Minkowski plane with quadratic form \( x_1^2 - x_2^2 \), will show \( \Psi_{aff} \) to be the hyperboloid of one sheet \( x_1^2 - x_2^2 + y^2 = 1 \) in \( V^+ \). In this case, the stereographic projection from \( V \) is not defined when \( x_1^2 - x_2^2 + 1 = 0 \).

Most hyperspheres and hyperplanes of \( V \) are carried by stereographic projection to the intersection of \( n \)-planes with \( \Psi_{aff} \) (as a matter of fact, all of them in the case of the classical Euclidean plane). Most of these \( n \)-planes correspond to their poles. In general these correspondences fall into the following four cases.

Any hypersphere \( (x-a)'(x-a) = 0 \) and, \( a \neq 0 \) or \( \alpha + \epsilon \rho^2 \neq 0 \), corresponds to \( n \)-plane \( a' \cdot x' = \epsilon \rho^2 \). Its pole is

\[
a' = a - \frac{2\epsilon \rho^2}{\alpha - \alpha + \epsilon \rho^2} + \frac{a - \alpha - \epsilon \rho^2}{\alpha - \alpha + \epsilon \rho^2}.
\]  

(3.1)

This \( n \)-plane of \( V^+ \) passes through neither \( \epsilon \rho \) nor 0, and its pole \( a' \) represents the hypersphere. The odd looking condition that \( a \neq 0 \) or \( \alpha + \epsilon \rho^2 \neq 0 \) is required in order to guarantee that the stereographic projection is defined on the points of the hypersphere. Observe that the excluded hypersphere \( (x - 0)'(x - 0) = -\epsilon \rho^2 \) does not even have an image under stereographic projection, although it may be considered to be represented by the point \( a' = 0 \) in \( V^+ \). This latter phenomenon does not occur for the classical Euclidean plane, but it is seen immediately for the Minkowski plane. To characterize the singular hyperspheres among these, note the identity

\[
a' \cdot a' = \rho^2 = \left( \frac{2\epsilon \rho^2}{\alpha - \alpha + \epsilon \rho^2} \right)^2.
\]
Then $\alpha$ vanishes when $a'$ is in $\Psi_{aff}$, that is, when the $n$-plane is tangent to $\Psi_{aff}$.

Any hypersphere $(x-a)(x-a) = \alpha$ with $a \cdot a - \alpha + \epsilon^2 = 0$ corresponds to the $n$-plane $a' \cdot z' = 0$, where $a' = a - \epsilon p$, passing through 0 but not through $\epsilon p$. The $n$-planes of such $n$-planes are at infinity. The hypersphere is singular when $a'$ is an isotropic vector in $V^+$.

A hyperplane $U \cdot (x - c) = 0$ with $U \cdot c \neq 0$ corresponds to the $n$-plane $a'\cdot z' = \epsilon p^2$ passing through $\epsilon p$ but not through 0. Its pole is

$$a' = \frac{U \cdot c}{U \cdot p} + \epsilon p.$$

The singular hyperplanes among these are characterized by the tangency of the $n$-plane of $V^+$ as a consequence of the formula

$$\frac{a' \cdot a' - \epsilon^2}{a \cdot a} = \frac{\epsilon^2}{U \cdot c} U \cdot u.$$

Finally, a hyperplane $U \cdot x = 0$ of $V$ corresponds to the $n$-plane $U \cdot z' = 0$ in $V^+$ passing through both $\epsilon p$ and 0. Its pole is at infinity. The singular hyperplanes are those for which $U$ is isotropic.

Among the $n$-planes in $V^+$ which meet $\Psi_{aff}$, the only one not considered above is the tangent $n$-plane with pole $\epsilon p$. This plane does not represent any hypersphere or hyperplane of $V$.

The correspondence — between: hyperspheres and hyperplanes of $V$, $n$-planes of $V^+$, and points in $V^+$ — has gaps in the general case, as observed above. These gaps can be overcome by the use of projective geometry.

3.2 To pass to the projective setting, we extend $V^+$ to a projective space $E_{n+1} = \mathbb{P}(W)$ containing a quadric $\Psi$ whose finite points are $\Psi_{aff}$. The quadric $\Psi$ is given by a quadratic form $(X|X)$, $X$ in $W$, unique up to a non-zero scalar factor. Take for $W$ the space $V + cK + fK$ consisting of vectors $X = x + cy + fz$, $x$ in $V$, $y$ and $z$ in $K$, and quadratic form

$$(X|X) = \lambda_0 (x \cdot x + cy^2 - \epsilon^2 z^2),$$

where $c$ and $\rho$ are the non-zero scalars to be related to $\Psi_{aff}$, and $\lambda_0$ is a fixed scalar not otherwise specified. The point $x + cy$ of $V^+$ lies in the finite part $E_{n+1} \setminus (f)^{1}$, with $x \neq 0$, gives homogeneous coordinates for the point $\frac{x + cy}{z}$ of $V^+$. Clearly $\Psi$ has $\Psi_{aff}$ as its finite points.

The map $j$ from $V$ to $\Psi$ given by

$$j(x) = \left( x + \epsilon \rho \frac{x \cdot x - \epsilon^2}{2\rho^2} + f \frac{x \cdot x + \epsilon^2}{2\rho^2} \right),$$

with inverse $j^{-1}\left((x + cy + fz) = x(\epsilon y + z)^{-1}\right)$, is a bijection between $V$ and $\Psi \setminus (\epsilon p + f)^{1}$. Via $j$, $\Psi$ is referred to as the "conformal compactification" of $V$ with $\Psi \setminus (\epsilon p + f)^{1}$ being the "cone at infinity". The map $j$ restricted to $\Psi \setminus (\epsilon p + f)^{1}$ is the composition of stereographic projection to $\Psi_{aff}$ followed by the inclusion of $V^+$ in $E_{n+1}$.

Next we establish the important connection between hyperspheres and hyperplanes of $V$ and sections of $\Psi$ by primes of $E_{n+1}$.

Let $\langle A \rangle = (a + \epsilon \rho b + f c)$ in $E_{n+1}$ be a point such that its polar prime $\langle A \rangle^1$ given by $(A|X) = 0$ meets $\Psi$. The $j^{-1}(\langle A \rangle \cap \Psi)$ has equation

$$x \cdot x - \frac{b + c}{2} + a \cdot x - \frac{b + c}{2} = 0$$

in $V$.

If $b \neq c$, that is, $\langle A \rangle$ is not on $(\epsilon p + f)^{1}$, then completing squares gives

$$\left( x - \frac{a}{-b + c} \right) \left( x - \frac{a}{-b + c} \right) = \frac{a \cdot a + \epsilon^2 b^2 - \epsilon^2 c^2}{(-b + c)^2},$$

and we have a hypersphere with center $a(-b + c)^{-1}$ and square-radius $\frac{a \cdot a + \epsilon^2 b^2 - \epsilon^2 c^2}{(-b + c)^2}$. This square-radius can also be written

$$\frac{(A|A)(\epsilon p|\epsilon p) \epsilon^2}{(A|\epsilon p + f)^2}.$$

Remark. This formula is the bridge between square-class of hyperspheres and square-class of points on $E_{n+1}$ as they will be defined shortly in terms of the numerator.

If $b = c$, that is, $\langle A \rangle$ lies on $(\epsilon p + f)^{1}$, the equation is $a \cdot x = \epsilon^2 \frac{b + c}{2}$, and we have a hyperplane of $V$. (The exception is when $a$ is zero and $b = c$ is not zero, and then $\langle A \rangle$ is $(\epsilon p + f)$.) Observe that in the case of a hyperplane the numerator of $(*)$ is $(a \cdot a)(\epsilon^2)$ up to a square.

In fact, we obtain all the hyperspheres and hyperplanes of $V$ by pulling back by $j$ sections of $\Psi$ by primes. The hypersphere $(x - a) \cdot (x - a) = \alpha$ is obtained from

$$\langle A \rangle = \left( a + \epsilon \rho \frac{a \cdot a - \epsilon^2}{2\rho^2} + f \frac{a \cdot a - \epsilon^2 + \epsilon^2}{2\rho^2} \right),$$

which is the pole of the required prime. Similarly, the hyperplane $U \cdot (x - c) = 0$ is obtained from

$$\langle A \rangle = \left( u + \epsilon \rho f \frac{u \cdot u}{\epsilon \rho^2} \right).$$

These expressions are the projective extensions of $(3.1)$ and $(3.2)$. In contrast to the affine case, they are not restricted to certain hyperspheres and hyperplanes.

Thus, hyperspheres and hyperplanes of $V$, together with the cone at infinity, are represented by points $\langle A \rangle$ in $E_{n+1}$ such that $\Psi \cap (A)^{1}$ is not empty. Equivalently, by $(2.1)$, by points $\langle A \rangle$ such that $\psi_\alpha$ contains points other than $\langle A \rangle$. We denote the set of such points by $\Psi^*$ and call its elements spheres.

Spheres of $\Psi \subset \Psi^*$ represent singular hyperspheres, and singular hyperplanes of $V$, and the cone at infinity. These will be called singular spheres. Note, if $\langle A \rangle$ in $\Psi$ is a singular sphere, then $\nabla \alpha = \langle A \rangle^{1}$ is contained in $\Psi^*$. For, if $\langle X \rangle \neq \langle A \rangle$ lies in $\langle A \rangle^{1}$, then $(X|A)^2 - (X|X)(A|A) = 0$ gives $\langle A \rangle \in \psi_\alpha$, and hence $\langle A \rangle$ is in $\Psi^*$.

Formula $(*)$ also reflects the two steps to make a hypersphere correspond to a point in projective space $E_{n+1}$. 

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Namely, the stereographic projection into a quadric $Ψ_\text{aff}$ in $V^+$ centered at the origin with specified square radius $ε^2$, and the embedding of $V^+$ into $P^{n+1}$ taking $Ψ_\text{aff}$ into a projective quadric $Ψ$ defined by a quadratic form $(X | X)$ specified up to scaling. In the classical case, these square radii and scale factors are chosen to be 1 for convenience, but at the loss of making their roles transparent, especially in the following definition.

**3.4 The square-class of a point $(P)$ in $P^{n+1}$ not lying on $Ψ : x \cdot x + ε^2 - ε^2 z^2 = 0$ is the square-class of $(P | P)(ε^2 | ε^2) \in K$. This definition involves the choice of $ε$ in $K$ as does $Ψ$ itself. It is important to observe that this definition of square-class in $P^{n+1}$ does not depend on the scaling $λ_0$ of the quadratic form or the homogeneous coordinates of the points $(P)$ and $(ε^2)$. It is compatible with the notion of “of the same square-class” of Section 2.

From the numerator of $(*)$ it is clear that the square-class of $(A)$ in $Ψ^+ \setminus Ψ$ is the same as that of the non-singular hypersphere or hyperplane it represents. We summarize this as:

**Theorem 3.1.**

(a) The map which sends $(P)$ to

$$j^{-1}\left(\left(Ψ \setminus (ε^2 \cdot P)\right) \cap \{P\}\right) = \{x \in V \mid \left(x + ε^2 \cdot x - ε^2 \cdot f \cdot x + ε^2 \right) = 0\}$$

is a bijection from $Ψ^+ \setminus Ψ$ to the set of hyperspheres, hyperplanes, and the cone at infinity of $V$.

(b) Points of $Ψ \subset Ψ^+$ correspond to singular hyperspheres, singular hyperplanes, and the cone at infinity. A point $(P)$ of $Ψ^+ \setminus Ψ$ corresponds to a non-singular hypersphere or a non-singular hyperplane of $V$ whose square-class is that of the point $(P)$.

A point $(P)$ in $P^{n+1}$ is called representable if $(P | P)(ε^2 | ε^2) \in K$ is representable by the quadratic form $x \cdot x + f \cdot x$. (See Section 1.) Thus, $(A)$ is a sphere if and only if $(A)$ is representable as a point of $P^{n+1}$. Clearly two points of $P^{n+1} \setminus Ψ$ lying in the same square-class are either both representable or both not representable.

Consequently, $Ψ^+$ is the disjoint union of $Ψ$ and sets consisting of spheres having the same (representable) square-class.

**3.5 Some examples are in order since the classical Euclidean intuition is misleading, especially in regard to the cones and square-classes that play a fundamental role in the next section of this paper.**

**Examples:**

(f) Let $V$ be the real Euclidean plane. Using $ε = 1$ and $f = 1$, we obtain the quadric $Ψ : x^2 + x^2 + y^2 - z^2 = 0$ in $P^3$. Then $P^3$ consists of the quadric $Ψ$ and the points in the square-classes of 1 and $-1$. The set $Ψ^+$ of representable points of $P^3$, consists of $Ψ$ and the points in the square-class of 1. Given two non-singular spheres $(A)$ and $(B)$, their cones $Ψ_A$ and $Ψ_B$ always meet (in fact in two conics which are the intersection of either cone by two planes — see Section 4).

(g) Let $V$ be the “rational Euclidean plane”. As before $ε = 1$ and $f = 1$ and the quadric $Ψ$ is $x^2 + x^2 + y^2 - z^2 = 0$. Now $P^3 \setminus Ψ$ breaks up into many square-classes (see Section 1.1), and $Ψ^+$ consists of $Ψ$ and points belonging to infinitely many square-classes.

If two non-singular spheres are of the same square-class, their cones meet, as in the previous example. However, if they lie in different square-classes, one only knows that the intersection of their cones lies on $Ψ$ (see Section 2.2). In this case (spheres of different square-classes in the rational Euclidean plane) the intersection of two hyperspheres, as well as their corresponding cones, may be non-empty or empty. Both situations may occur.

As, for example, the cones corresponding to the hyperspheres $x^2 + z^2 = 1$ and $x^2 + z^2 = 2$ do not meet. The other possibility is more interesting.

Consider the non-singular hyperspheres $(x^2 - 1)(x^2 - 1) = a_1^2 a_2^2 - a_1^2 - a_2^2 - 1$ is a square. If we take $a_1 = 3$ and $a_2 = 3/2$ we have two hyperspheres in the different representable square-classes $8 = 2^3 + 2^3$ and $5/4 = 1^2 + (1/2)^2$ and having the points $(1, 2)$ and $(1/3, 2/3)$ in common.

A general hypersphere $(x^2 - 1)^2 + (x^2 - 2)^2 = α$ is represented by $(A)$ in $Ψ^+$.

$$(A) = (a_1, a_2) + (a_1^2 + a_2^2 - α + 1) + (a_1^2 + a_2^2 - α + 1)$$

The hyperspheres of the previous paragraph are represented by $(a_1, 0) + f$ and $(0, a_2) + f$, respectively. The cones with vertices at these points are

$$(a_1^2 - 1)(x^2 + y^2 - z^2) = (a_1 x_1 - z)^2 = 0$$

and

$$(a_2^2 - 1)(x^2 + y^2 - z^2) = (a_2 x_2 - z)^2 = 0$$

If a point $(x^2 + y^2 + f x)$ common to these cones were not to lie on $Ψ$ then $(a_1^2 - 1)/(a_2^2 - 1)$ would be a square, but the hyperspheres are assumed to be in different square-classes. Consequently, from $a_1 x_1 - z = 0$ and $a_2 x_2 - z = 0$, we obtain the intersection points of the two cones to be

$$(a_2, a_1) + (a_1^2 a_2^2 - a_1^2 - a_2^2 + 1),$$

which are the images under $j$ of the intersection of the two hyperspheres computed earlier.
We remark that if the usual topology of $P^3(K)$ were restricted to $P^3(Q)$, every open set would contain points of infinitely many square-classes. Even so, for any point $(A)$ all the points in $\psi_A \setminus \Psi$ are of the same square-class. In particular, along any generator of $\psi_A$, all points, excluding those lying on $\Psi$, lie in the square-class of $(A)$. So, for $(B)$ in a distinct square-class from that of $(A)$, a generator of the cone $\psi_B$ could possibly meet a generator of $\psi_A$ only on the quadric.

Example:

(1) Let $V$ be a Minkowski plane over a field $K$. Using the fact that every $\alpha$ in $K$ can be written as $(\alpha + 1)/2 - (\alpha - 1)/2$, it is easy to show that every $(P)$ in $P^3$ (belonging to any square-class) is representable. Hence $\Psi = P^3$. Again, as in Example (f), if the non-singular spheres $(A)$ and $(B)$ are in the same square-class, then $\psi_A \cap \psi_B$ is not empty.

In general, if $(A)$ and $(B)$ lie in different square-classes, it follows from our earlier remarks in this section and equation (2.1) that

$$\psi_A \cap \psi_B = \Psi \cap \psi_A \cap \psi_B = \Psi \cap (A)^\perp \cap (B)^\perp.$$ 

Hence $\psi_A \cap \psi_B$ is empty or not, according as $(A, B)^\perp = (A)^\perp \cap (B)^\perp$ meets the quadric $\Psi$ or not.

3.6 A pencil of hyperspheres, as studied in Section 1, gives rise to a line in $P^{n+1}$, as suggested by Section 1.4. More precisely, given two different hyperspheres $(x-a) \cdot (x-a) = \alpha$ and $(x-b) \cdot (x-b) = \beta$, denote by $(A, B)$ the line in $P^{n+1}$ passing through the points

$$(A) = \left( a + \epsilon \rho a \cdot a - \alpha - \epsilon \rho^2 \over 2 \epsilon \rho^2 \right) + f a \cdot a - \alpha + \epsilon \rho^2$$

and

$$(B) = \left( b + \epsilon \rho b \cdot b - \beta - \epsilon \rho^2 \over 2 \epsilon \rho^2 \right) + f b \cdot b - \beta + \epsilon \rho^2$$

Thus, the set of hyperspheres of the pencil determined by the two given hyperspheres corresponds precisely to the set of points of

$$\Psi^+ \setminus \{ (\epsilon \rho + f)^\perp \} \cap (A, B).$$

Such lines passing through $(\epsilon \rho + f)^\perp$ correspond to pencils of concentric hyperspheres with the cone at infinity joined.

Since points of $(\epsilon \rho + f)^\perp \setminus \{ (\epsilon \rho + f) \}$ in $\Psi^+$ represent precisely hyperplanes of $V$, lines lying in this set will correspond to pencils of hyperplanes in $V$, as are defined in a fashion similar to pencils of hyperspheres (see Section 1.4). Lines in $(\epsilon \rho + f)^\perp$ passing through $(\epsilon \rho + f)$ represent pencils of parallel hyperplanes with the cone at infinity joined.

In the above spirit we can say that pencils of hyperplanes or hyperspheres in $V$ correspond to lines in $P^{n+1}$. In fact, this correspondence is deeper.

Given two hyperspheres $(x-a) \cdot (x-a) = \alpha$ and $(x-b) \cdot (x-b) = \beta$ for which $\delta = (a-b) \cdot (a-b)$ is not zero, the discriminant $\Delta$ of the quadratic equation

$$\alpha(1-t) + \beta t - \delta t(1-t) = 0 \quad (3.3)$$

(see Section 1.4) is of the same square-class as

$$(A|B)^2 - (A|A)(B|B) = \frac{\lambda^2_0}{4} ((\delta - \alpha - \beta - 4 \alpha \beta) \quad (3.4)$$

where $(A)$ and $(B)$ represent the given hyperspheres and $\lambda_0$ is the unspecified fixed scalar for the quadratic form $(X|X)$ on $W$. This is a consequence of a simple calculation using

$$\left( a + \epsilon \rho a \cdot a - \alpha - \epsilon \rho^2 \over 2 \epsilon \rho^2 \right) + f a \cdot a - \alpha + \epsilon \rho^2$$

and

$$\left( b + \epsilon \rho b \cdot b - \beta - \epsilon \rho^2 \over 2 \epsilon \rho^2 \right) + f b \cdot b - \beta + \epsilon \rho^2$$

is

$$= \frac{\lambda_0}{2} ((a-b) \cdot (a-b) - \alpha - \beta) \quad (3.4)$$

The line $(A, B)$ meets the quadric $\Psi$ in two, or no points, as the expression (3.4) is a non-zero square, zero, or a non-zero square, respectively.

Hence, in view of Section 1.4, the pencil of hyperspheres determined by the hyperspheres $(x-a) \cdot (x-a) = \alpha$ and $(x-b) \cdot (x-b) = \beta$ (for which $\delta$ is not zero) contains two, one, or no singular hyperspheres, as the corresponding line in $P^{n+1}$ meets the quadric in two, one, or no points. Moreover, the singular hyperspheres of the pencil correspond exactly to the points where the line meets $\Psi$.

Given two hyperspheres for which $\delta$ is zero, we obtain that (3.3) is not quadratic, so $\Delta$ does not account for the number of solutions, that is, the number of singular hyperspheres in the pencil. Although, still the corresponding line in $P^{n+1}$ meets $\Psi$ in two, one, or no points as the pencil contains two, one, or no "singular objects". We mean by "singular objects" either singular hyperspheres, singular hyperplanes, or the cone at infinity. A pencil of hyperspheres consists of singular hyperspheres in $V$ if and only if the corresponding line lies in the quadric $\Psi$.

Analogous statements hold for pencils of hyperplanes.

3.7 The group $PO(W)$, introduced in Section 2, sends lines of $P^{n+1} = P(W)$ to lines. It is of interest to interpret this in terms of the geometry of pencils of hyperspheres and hyperplanes of $V$. For brevity we call these simply pencils. We focus especially on the square-classes.

Let $A$ be a non-isotropic vector of $W$. The transformation of $W$ given by

$$X \sim X' = X - A^\perp (A|X) \quad (3.5)$$

which is the reflection in the non-singular subspace $(A)^\perp$, gives a transformation of $P^{n+1}$ which fixes $(A)$ and $(A)^\perp$ pointwise, and sends $(X)$ to the point $(X')$ for which

$$\{ (X'), (X); (A), (A, X) \cap (A)^\perp \}$$
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is a harmonic tetrad. In particular, if \( (X) \) lies on \( \Psi \), then \( \langle X' \rangle \) on \( \Psi \) is determined by \( \langle A \rangle, (X), \) and \( \langle X' \rangle \) being collinear.

Suppose, for the moment, that

\[
\langle A \rangle = \left\langle a + \epsilon \rho \frac{a \cdot a - \alpha - \epsilon \rho^2}{2\epsilon \rho^2} + f \frac{a \cdot a - \alpha + \epsilon \rho^2}{2\epsilon \rho^2} \right\rangle
\]

is in \( \mathcal{P} \\| (\epsilon \rho + f) \). We do not assume that \( \alpha \) is representable; that is, \( \langle A \rangle \) need not to be a sphere. Using the inclusion \( j : V \rightarrow \Psi \) given by

\[
j(x) = \left\langle x + \epsilon \rho \frac{x \cdot x - \epsilon \rho^2}{2\epsilon \rho^2} + f \frac{x \cdot x + \epsilon \rho^2}{2\epsilon \rho^2} \right\rangle,
\]

it is easy to check that \( \langle A \rangle, j(x), \) and \( j(x') \) are collinear exactly when \( a, x, \) and \( x' \) are collinear and \( \langle x'-a \rangle \cdot \langle x-a \rangle = \alpha \) \ldots Hence this transformation of \( \mathcal{P} \), which sends \( \Psi \) to itself, “pulls back” to \( V \) as the transformation

\[
x' = a + (x-a) \frac{\alpha}{(x-a) \cdot (x-a)}
\]

(3.6)

which, although not everywhere defined, is a birational transformation of \( V \). Classically this is called inversion with center \( a \) and power \( \alpha \). If \( \langle A \rangle \) lies in \( \epsilon \rho + f \), one obtains in a similar fashion reflections in hyperplanes of \( V \).

The Cartan–Dieudonné theorem [16, p. 27] establishes that the reflections (3.5) generate \( \mathfrak{O}(W) \), so the resulting transformations of \( \mathcal{P} \) generate \( \text{PO}(W) \). Classically, one says that inversions and reflections in hyperplanes generate the Möbius group or the inverse group \( \mathcal{M}(V) \) of birational transformations of \( V \) (actually, inversions suffice). This description of \( \mathcal{M}(V) \) by \( \text{PO}(W) \) was in fact known to F. Klein and his contemporaries at the turn of the nineteenth century [9, p. 195].

The inversion (3.6) has fixed points exactly when \( \alpha \) is representable by the quadratic form of \( V \). In this case, the set of fixed points is the hypersphere \( (x - a) \cdot (x - a) = \alpha \). One classically refers to such an inversion as an inversion in a hypersphere. Inversions in hyperspheres generate a subgroup of \( \mathcal{M}(W) \) or, more properly, transformations of \( \Psi \) arising from reflection (3.5) for which the square–class of \( \langle A \rangle \) is representable. This subgroup of \( \text{PO}(W) \) will be denoted \( \text{PO}(W)_{\text{rep}} \).

It is easy to check that \( \text{PO}(W)_{\text{rep}} \) is all of \( \text{PO}(W) \) in case that the field \( F \) is \( \mathbb{R} \), or any algebraically closed field, or if the quadratic form \( x \cdot x \) on \( V \) has isotropic vectors. It may actually be a proper subgroup. This is seen in the case for the rational Euclidean plane from the inversion with center \( (0, u) \) and power 3. (This is a consequence of 3 mod \( (Q^*)^2 \) not being in the image of \( \text{PO}(W)_{\text{rep}} \) under the spinorial norm from \( \text{PO}(W) \) to \( Q^*/(Q^*)^2 \), when the choices are \( \epsilon = 1, \rho = 1, \) and \( \lambda_0 = 1 \).) We do not pursue further here the characterization of \( \text{PO}(W)_{\text{rep}} \) as a subgroup of \( \text{PO}(W) \) in terms of quadratic forms.

3.8 We now list invariants in \( V \) which define the orbits of the action of \( \mathcal{M}(V) \) on the pencils of \( V \).

It follows directly from Section 2.3 that:

- Two pencils consisting of singular hyperspheres or hyperplanes lie in the same orbit under \( \mathcal{M}(V) \).
- Two pencils each containing exactly two singular hyperspheres or hyperplanes lie in the same orbit.
- Two pencils that contain precisely one singular hypersphere or hyperplane are in the same orbit under \( \mathcal{M}(V) \) if and only if there are two hyperspheres, a hypersphere and hyperplane, or two hyperplanes, one in each pencil, which are of the same square–class.
- For two pencils without singular hyperspheres or hyperplanes we need to introduce additional invariants.

First consider a pencil of hyperspheres which does not contain any singular ones. Let \( (x - a) \cdot (x - a) = \alpha \) and \( (x - b) \cdot (x - b) = \beta \) be two different hyperspheres in the pencil. Since this pencil does not contain any singular objects, \( \epsilon = (b - a) \cdot (b - a) \) is not zero, for otherwise the discriminant \( \Delta = (b - a) \cdot (b - a) = 4 \alpha \beta \) would be a square. Hence the radical trace \( m \) is well defined. Define \( \omega = (m - a) \cdot (m - a) - \alpha \) to be the power of the radical trace with respect to \( (x - a) \cdot (x - a) = \alpha \). Notice that the power of \( m \) with respect to any of the hyperspheres of the pencil is precisely \( \omega \).

A computation shows that \( \omega = \frac{\Delta}{4 \epsilon} \). If the intersection of the hyperspheres \( (x - a) \cdot (x - a) = \alpha \) and \( (x - b) \cdot (x - b) = \beta \) happens to be non-empty, it will be a hypersphere of the radical hyperplane of the pencil having center \( m \) and square radius \( -\omega \). So the expression \( \omega/\delta \) is affinely meaningful and it is easy to check that its square–class depends only on the pencil. This square–class will be called the ratio–invariant of the pencil. The ratio–invariant coincides with the square–class of the negative of the Gram determinant of the line \( \langle A, B \rangle \).

If the given pencil is a pencil of hyperplanes, then the expression \( \omega/\delta \) may not be defined. But even in this case we may associate a “ratio–invariant”, namely the square–class of the negative of the Gram determinant of the corresponding line to the pencil.

Now using the fact that the square–class of a sphere in \( \Psi^+ \) coincides with the square–class of the hypersphere or hyperplane it represents, we obtain the following:

Two pencils, without singular objects, are in the same orbit under the action of the Möbius group if and only if their ratio–invariants are equal, and there are hyperspheres or hyperplanes, one in each pencil, which have a common square–class.

4. THE GENERALIZED APOLLONIUS CONTACT PROBLEM

4.1 The classical Apollonius problem in the real Euclidean plane consists of finding all circles which are tangent to three given circles.

The classical notions of incidence and tangency of inverse geometry in the Euclidean plane may now be treated in terms of incidence and tangency in projective space \( \mathcal{P}^3 \) with the quadric \( \Psi \), as set out in Section 3. By representing the circles of the Euclidean plane as points in \( \Psi^+ \) we lose
the notion of radius and the ability to distinguish between lines and circles. But, this does not affect the treatment of problems from inersive geometry, because radius is not an invariable.

In general, placing inersive problems in the context of the geometry of \( \Psi^+ \) allows one to treat the various hyperspheres and hyperplanes in a unified manner. It is in this setting then, that of the geometry of \( \Psi^+ \), that we will consider the generalised Apollonius problem.

Recall that for a sphere \( \langle A \rangle \) in \( \Psi^+ \), the cone \( \psi_A \), with vertex at \( A \), consists of those \( X \) such that the line \( \langle A, X \rangle \) is tangent to \( \Psi \). Two spheres \( \langle A \rangle \) and \( \langle B \rangle \) are called tangent when \( \langle A, B \rangle \) is a line tangent to \( \Psi \) or, equivalently, if one sphere lies on the cone with vertex at the other. This concept of tangency extends the notion of tangency for hyperspheres in \( \Psi \), as defined at the beginning of Section 1.5 by \((1.2)\). This is a consequence of \((3.4)\) in Section 3.6.

4.2 Let \( V \) be the affine space of Section 1. Given \( n+1 \) "independent" hyperspheres in \( V \), the generalized Apollonius problem consists of finding the hyperspheres and hyperplanes which are tangent to the given ones.

In the representation of Section 3, the above problem becomes the following. Given \( n+1 \) independent hyperspheres \( \langle A_1 \rangle, \ldots, \langle A_{n+1} \rangle \) (that is, \( n+1 \) points of \( \Psi^+ \) which lie in a unique \( n \)-flat of \( \mathbb{P}^{n+1} \)), determine all those spheres of \( \Psi^+ \) each of which is tangent to all of the \( \langle A_i \rangle \). If we denote by \( \psi_1, \ldots, \psi_{n+1} \) the cones with vertices at these points, then the spheres which are solutions to the Apollonius problem are exactly those of the intersection \( \bigcap_{i=1}^{n+1} \psi_i \).

In the original affine setting this projective view includes the possibility that some of the spheres \( \langle A_i \rangle \) are singular or represent hyperplanes in \( V \), with appropriate affine notions of tangency.

4.3 First we consider the Apollonius problem when all of the \( n+1 \) given spheres are non-singular. Furthermore, we assume that they lie in the same square-class, for otherwise all the solutions to the problem would be singular, as shown in Section 2.2.

Let \( \langle A \rangle \) and \( \langle B \rangle \) be two non-singular spheres of the same square-class, that is, \( (\psi | B) = \lambda^2 (A | A) \) as in Section 2.1, where \( \lambda \) is a non-zero element of \( \mathbb{K} \). Define the anti-similitude points \( (K) \) and \( (K') \) by \( K = B - A \lambda \) and \( K' = B + A \lambda \). In fact, the pair of points \( (K) \) and \( (K') \) are determined as the conjugate points on the line \( \langle A, B \rangle \) for which \( \langle A \rangle, \langle B \rangle, \langle K \rangle, \langle K' \rangle \) is a harmonic tetradi.

As in the classical case, if an anti-similitude point represents a hypersphere, then inversion in that hypersphere interchanges the hyperspheres or hyperplanes corresponding to \( \langle A \rangle \) and \( \langle B \rangle \). (Cf. the Proposition 2.1.)

For \( \lambda \) as above, we have the identity

\[
(B | X)^2 - (B | B) (X | X) = (A | X)^2 - (A | A) (X | X) \lambda^2
= (B - A \lambda | X) (B + A \lambda | X),
\]

from which it follows that

\[
\psi_A \cap \psi_B = \left( \psi_A \cap (K)^\perp \right) \cup \left( \psi_A \cap (K')^\perp \right).
\]

This will be used inductively on \( A_1, A_2, \ldots, A_{n+1} \).

Let \( (K_{ij}) \) and \( (K'_{ij}) \) be the anti-similitude points determined by \( (A_i) \) and \( (A_j) \); there are \((\frac{n+1}{2})\) such pairs of points.

The points \( (K_{12}), \ldots, (K_{n,n+1}) \) determine an \( (n-1) \)-flat \( L = (K_{12}, \ldots, K_{n,n+1}) \) because \( (A_1), \ldots, (A_{n+1}) \) determine an \( n \)-flat. Likewise, the points obtained by replacing any number of the unprimed anti-similitude points by the corresponding primed ones also span an \((n-1)\)-flat. There are \(2^n\) of these \((n-1)\)-flats in \( \mathbb{P}^{n+1} \); they will be called anti-similitude secundae and denoted by \( L^- \).

The induction now gives that, for the \( n+1 \) conics \( \psi_1, \ldots, \psi_{n+1} \), we have

\[
\bigcap_{i=1}^{n+1} \psi_i = \bigcup_{p=1}^{2^n} \left( \psi_1 \cap L^+_p \right),
\]

where the \( L^+_p \) range over the conjugates with respect to \( \Psi \) of the anti-similitude secundae. Each \( L^+_p \) is a line in \( \mathbb{P}^{n+1} \).

4.4 In the extreme case that all the spheres \( \langle A_1 \rangle, \ldots, \langle A_{n+1} \rangle \) are singular, each cone \( \psi_i \) with vertex \( (A_i) \) coincides with the polar prime \( (A_i)^\perp \) and \( \bigcap_{i=1}^{n+1} \psi_i = \bigcup_{i=1}^{n+1} (A_i)^\perp \) is just one sphere.

Classically, in this extreme case for Euclidean \( n \)-dimensional space, since the singular hyperspheres are points or point-spheres, the Apollonius problem has only one solution and that solution is the hypersphere passing through the \( n+1 \) given points.

4.5 Now, among the given \( n+1 \) spheres suppose that we have \( n - k \) singular spheres and \( k + 1 \) non-singular spheres, the latter all of the same square-class. We assume, for the moment, that \( k \geq 1 \). Let \( (A_1), \ldots, (A_{n+1}) \) be non-singular and \( (A_{k+2}), \ldots, (A_{n+1}) \) singular, and let \( (K_{ij}) \) and \( (K'_{ij}) \) be the anti-similitude points determined by \( (A_i) \) and \( (A_j) \) for \( i, j = 1, \ldots, k + 1 \). There are \( 2^k \) anti-similitude flats, each of dimension \( k - 1 \). They are of the form \( L = (K_{12}, K_{23}, \ldots, K_{k+1}k+1} \) or are obtained from this by replacing \( K \)'s by \( K' \)'s, as before. The intersection of the conics \( \bigcap_{i=1}^{n+1} \psi_i \) now becomes

\[
\bigcup_{s=1}^{2^k} \psi_1 \cap \left( L^-_s \cap (A_{k+2}, \ldots, A_{n+1})^\perp \right),
\]

where the \( L^-_s \) range over the anti-similitude \((k-1)\)-flats. Each \( L^-_s \cap (A_{k+2}, \ldots, A_{n+1})^\perp \) is a line of \( \mathbb{P}^{n+1} \).

In case there is only one non-singular sphere, say \( (A_1) \), and \( k = 0 \), the above formula survives in the trivial form \( \psi_1 \cap (A_2, A_3, \ldots, A_{n+1})^\perp \).

4.6 Finally, we consider the case when among the \( n+1 \) given spheres, in addition to the singular ones, there are spheres of more than one square-class. That is, the most general Apollonius problem is the case that the given \( n+1 \) spheres consist of \( k+1 \) non-singular spheres lying in \( m \) distinct square-classes and of \( n-k \) singular spheres.
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For \( 1 \leq h \leq m \), let \( k_h + 1 \) be the number of spheres in the \( h \)-th square-class, so \( \sum_{h=1}^m (k_h + 1) = n + 1 \). Changing slightly the indexing of our earlier notation, we denote the spheres of the \( h \)-th square-class by \( \langle A_h \rangle \), \( i = 1, 2, \ldots, k_h + 1 \). If \( \langle A_h \rangle \) is another sphere in the same square-class, we denote by \( \langle K_i \rangle \) and \( \langle K_j \rangle \) the anti-similitude points determined by them.

If the \( h \)-th square-class contains \( k_h + 1 \geq 2 \) spheres, then one constructs anti-similitude flats of the form \( L^h = \{ K_{i} : i = 1, 2, \ldots, k_h \} \) or by substituting any of the \( K \)'s by \( K_i \)'s. If \( \psi_i \) denotes the cone with vertex at \( \langle A_i \rangle \), then the intersection of the cones corresponding to the \( h \)-th square-class is given by

\[
\bigcap_{i=1}^{k_h+1} \psi_i = \psi_i \cap \bigcup_{i=1}^{k_h+1} (L^h)^{\perp},
\]

where the union ranges over the \( 2^{k_h} \) anti-similitude flats \( L^h \). Each \( L^h \) is a \( (k_h - 1) \)-dimensional flat.

As before, if \( k_h = 0 \) there is no anti-similitude flat defined, and the above formula holds by taking \( (L^h)^{\perp} \) to mean all of \( P^{n+1} \).

The intersection of all the cones corresponding to the non-singular spheres among the \( n + 1 \) given spheres is

\[
\bigcap_{i=1}^{m} \psi_i = \left( \bigcap_{i=1}^{m} \psi_i \right) \cap \left( \bigcup (L^1)^{\perp} \cap (L^2)^{\perp} \cap \cdots \cap (L^m)^{\perp} \right),
\]

(4.2)

where the intersection on the left hand side is for \( i = 1, 2, \ldots, k_h + 1 \) and \( h = 1, 2, \ldots, m \); and the union on the right hand side ranges over all anti-similitude flats \( L^j \) of each square-class, for \( j = 1, 2, \ldots, m \). Note that (4.2) reduces to (4.1) when the given \( n + 1 \) spheres are non-singular and lie in the same square-class.

Recalling Section 3, where we proved that \( \psi_A \cap \psi_B \subset \Psi \) for two non-singular spheres \( (A) \) and \( (B) \) lying in distinct square-class, we have

\[
\psi_A \cap \psi_B = \Psi \cap \langle A, B \rangle^{\perp}.
\]

So, by induction, it follows that

\[
\bigcap_{h=1}^{m} \psi_i = \psi_i \cap \left( \bigcap_{i=1}^{m} (L^1)^{\perp} \cap (L^2)^{\perp} \cap \cdots \cap (L^m)^{\perp} \right),
\]

(4.3)

where the intersection on the left and the union of the right hand side range as specified for (4.2).

The solution of the Apollonius problem will be given by the intersection of the cones of all the \( n + 1 \) spheres. If \( m \), the number of distinct square-classes, is greater than 1, then this intersection may be written

\[
\bigcap_{i=1}^{m} \left( \psi_i \cap (A_1^1, A_1^2, \ldots, A_1^m)^{\perp} \right) \cap \cdots \cap \left( L^1 \right)^{\perp} \cap \cdots \cap \left( L^m \right)^{\perp} \cap \cdots \cap \left( \bigcap_{i=1}^{m} (A_{j+1}^1, \ldots, A_{j+1}^m)^{\perp} \right),
\]

where the union is taken over all anti-similitude flats \( L^j \) of each square-class for \( j = 1, \ldots, m \); \( \langle A_1^1 \rangle, \langle A_1^2 \rangle, \ldots, \langle A_1^m \rangle \) are choices of fixed spheres, one in each square-class, and \( \langle A_{j+2}^1, \ldots, A_{j+1}^m \rangle \) are the singular spheres.

It follows from the independence of the \( n + 1 \) given spheres that in the previous formula each intersection

\[
\left( L^1 \right)^{\perp} \cap \cdots \cap \left( L^m \right)^{\perp} \cap \langle A_1 \rangle \cap \langle A_2 \rangle \cap \langle A_3 \rangle \cap \langle A_4 \rangle \cap \cdots \cap \langle A_{n+1} \rangle^{\perp}
\]

is a point in \( P^{n+1} \) or, more precisely, a sphere. It will be a solution of the Apollonius problem exactly when it lies on \( \Psi \).

4.7 Using the fact that the given \( n + 1 \) spheres are independent as points of \( P^{n+1} \), and the above expression, a counting argument shows that if the number \( m \) of distinct square-classes is greater than 1, then the maximum number of solutions is \( 2^{k_h+1} - k_h = 2^{k_h+1} - m \), where \( k_h + 1 \) is the number of non-singular spheres. This holds even if some \( k_h \) are 0.

In case \( m = 1 \), the maximum number of solutions to the Apollonius problem is \( 2^{k_h+1} \). This can be seen by counting the anti-similitude flats when \( k > 0 \), and noting that with one exception the cones are primes when \( k = 0 \). This is the generalization of the fact that the maximum number of solutions to the Apollonius problem in the classical case of three independent spheres (circles) in the Euclidean plane is 2, 4, or 8 as two, one, or none of the given circles are singular. See Bruen et al. [2] and Muirhead [12].

One can now count in general the number of solutions to an Apollonius configuration by knowing how to determine the number of intersections that a line has with a cone.

For \( (A) \) not in \( \Psi \), the line through the points \( (Q) \) and \( (R) \) not lying on the cone \( \psi_A \) meets this cone in two, one, or no points according as the determinant of the Gram matrix

\[
D = \begin{bmatrix}
(A|A) & (A|Q) & (A|R) \\
(Q|A) & (Q|Q) & (Q|R) \\
(R|A) & (R|Q) & (R|R)
\end{bmatrix}
\]

of the quadratic form of \( W \), restricted to the subspace \( (A,R,Q) \), is the square-class of \( -(A|A) \), is zero, or neither. For the proof, it is enough to observe that the equation

\[
(Qs + Rt|A)^2 - (Qs + Rt|Qs + Rt) (A|A) = 0
\]

has two, one, or zero solutions in the field \( K \), according as the discriminant

\[
4[(Q|A)(R|A)-(Q|R)(A|A)] - 4[(Q|A)^2 - (Q|Q)(A|A)] \times \\
[(R^2 - (R|R)(A|A)] = -4(A|A) \det D
\]

is a non-zero square, is zero, or neither.

5. AN ALGORITHM

5.1 Knowing that in the absence of multiple square-classes, the solutions to the Apollonius problem lie on lines
which are conjugate to anti-similitude secunda, we obtain an algorithm that easily produces points which determine these lines.

We will assume throughout this section that the $n+1$ given spheres determine a prime of $P^{n+1}$ which is not tangent to the quadric $\Psi$.

Let $(A_1), \ldots, (A_{n+1})$ be the spheres which determine the prime. Suppose that $(A_1), \ldots, (A_{k+1})$ are non-singular and lie in the same square-class, and that $(A_{k+2}), \ldots, (A_{n+1})$ are singular.

5.2 Let $(Q)$ be the pole of the above prime and $(R)$ the intersection of one of the lines with $(Q)^\perp$. The points $(A_1)$, $(Q)$, and $(R)$ were used in (4.3) of Section 4.7 to determine the number of solutions of the Apollonius problem which are contributed by that line.

The point $(Q)$ is easily computed using, essentially, Gram's Rule. More specifically, with respect to some orthogonal basis of $W$, the $n+2$ components of $(Q)$ are the determinants, with appropriate signs, of the $(n+1) \times (n+1)$ submatrices of the matrix whose columns are the components of the $A_i$.

Assume for the moment that $k \geq 1$. Since there is only one square-class involved, by rescaling the vectors $A_i$, we may assume that $(A_1|A_2) = (A_1|A_i)$ for all $i = 1, 2, \ldots, k + 1$. Let

$$G = \left( A_1, \ldots, A_{n+1} \right) (A_1, \ldots, A_{n+1}) = (A_i|A_j)$$

be the $(n+1) \times (n+1)$ Gram matrix of the quadratic form of $W$ restricted to the non-singular subspace $(A_1, \ldots, A_{n+1})$. The basis

$$B_1, \ldots, B_{n+1} = (A_1, \ldots, A_{n+1}) G^{-1}$$

is dual to the basis $(A_1, \ldots, A_{n+1})$, that is, $(B_i|A_j) = \delta_{ij}$.

Now, the unprimed anti-similitude points are given by $A_1 - A_j$, and $(A_1 - A_2, A_2 - A_3, \ldots, A_k - A_{k+1})$ is an $k$-dimensional subspace of $W$ which corresponds to an anti-similitude $(k-1)$-flat of $P^{n+1}$. The other anti-similitude flats are obtained by replacing any number of the unprimed anti-similitude points by their corresponding primed ones $A_1 + A_j$. Hence, any of the $2^k$ anti-similitude flats can be specified by $e_1, e_2, \ldots, e_k$, each of which is $\pm 1$, as being spanned by the vectors $A_i - e_i A_i$. Now, it is clear that the vector

$$R = B_1 + e_1 B_2 + e_2 e_3 B_3 + e_1 e_2 e_3 B_4 + \cdots + e_1 e_2 \cdots e_k B_{k+1}$$

is orthogonal to the vectors of the subspace of $W$ specified by $e_1, e_2, \ldots, e_k$.

By computing, for all possible $e_i$, the square-class of the negative of the $3 \times 3$-determinant of (4.3) of Section 4.7, using the above choices of $A_1$, $Q$, and $R$, one thus obtains the number of solutions to the generalized Apollonius problem.

Observe that this calculation requires only the inversion of one matrix and the computation of a large $(n+2) \times (n+2)$ determinant just once. The repetitive steps use only three-by-three determinants.

In the case $k = 0$, that is, there is only one non-singular sphere, one may take the vector $R$ to be $B_1$.

Finally, if all of the $n+1$ spheres are singular, then the cones are tangent primes and their intersection is a point in $P^{n+1}$ which is a sphere, since every point of a tangent prime lies in $\Psi^+$, as noted in Section 3.3.

REFERENCES