The Formulas of Euler and Cagnoli

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Abstract

Formulas for the area of a spherical triangle date back centuries to well known names such as Euler, and to others not well known. Classical proofs typically relied on several uses of identities for spherical triangles. Here we will obtain, using elementary algebra, two formulas for the area of a spherical triangle due, respectively, to Euler and Cagnoli. These two formulas are very closely related. We will conclude with formulas suitable for calculations.

Let $A, B, C$ be the vertices of a spherical triangle lying in one hemisphere, $a, b, c$ the sides opposite the corresponding vertices. Also, $A, B, C$ will denote the angular measure, in radians, at these vertices. The sum $A + B + C$ is always greater than $\pi$. The sides are angles at the center of the sphere, and are measured in radians.

**Area** A \textit{lune} on a sphere has equal angles at its two diametrically opposite vertices.

$$\frac{\text{area of lune (steradians)}}{2\pi (\text{steradians})} = \frac{\text{angle at either vertex (radians)}}{\pi (\text{radians})}.$$  

Using six lunes, the proof of Girard’s Theorem yields that twice the area of triangle $ABC$ is: $2A + 2B + 2C - 2\pi$ (steradians). Dividing by 2, which is $2(\text{steradians})/(\text{radian})$ for lunes, gives the \textit{(angular) spherical excess} $A + B + C - \pi$ (radians). This will be denoted by $E$.

**Classical Formulas** In addition to the three angles of a spherical triangle, spherical excess may be expressed in terms of the three sides of a triangle, or a mixture of angles and sides. Here we consider only the area in terms of three sides.

**Euler** (1778)\textsuperscript{1}

$$\cos \frac{1}{2}E = \frac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}}$$

**Cagnoli** (ca. 1786)\textsuperscript{2}

$$\sin \frac{1}{2}E = \frac{2n}{4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}}$$

\textsuperscript{1}Leonhard Euler (1707-1783), Swiss mathematician, physicist, astronomer, logician, and engineer
\textsuperscript{2}Antoine Cagnoli (1743-1816), Italian astronomer.
where
\[ 4n^2 = 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c \]
(a classical notation).³

de Gua (ca. 1740s)⁴
\[ \tan \frac{1}{2} E = \frac{2n}{1 + \cos a + \cos b + \cos c} \]
which is the quotient of the two formulas above.

**Cosines and Sines** Everything devolves from the spherical Law of Cosines:
\[ \cos a = \cos b \cos c + \sin b \sin c \cos A \]
It will be convenient to write \( a \) for \( \cos a \), \( b \) for \( \cos b \), and \( c \) for \( \cos c \). There will be no confusion with the sides \( a, b, c \).

Beginning with the Law of Cosines
\[ \cos A = \frac{a - bc}{\sqrt{1 - b^2\sqrt{1 - c^2}}} \]
we have
\[ \sin^2 A = 1 - \left( \frac{a - bc}{\sqrt{1 - b^2\sqrt{1 - c^2}}} \right)^2 = \frac{(1 - b^2)(1 - c^2) - (a - bc)^2}{(1 - b^2)(1 - c^2)} = \frac{4n^2}{\sin^2 b \sin^2 c}, \]
so
\[ \sin A = \frac{2n}{\sqrt{1 - b^2\sqrt{1 - c^2}}} = \frac{2n}{\sin b \sin c}. \]
The quantities \( \sin B \) and \( \sin C \) are obtained by cyclically permuting \( A, B, C \) and \( a, b, c \). Dividing by \( \sqrt{1 - a^2} = \sin a \) and cyclically permuting gives the Law of Sines.

**Three angles** From
\[ \cos(A + B + C) = \cos A \cos B \cos C - \cos A \sin B \sin C \]
\[ - \sin A \cos B \sin C - \sin A \sin B \cos C \]
we have⁵
\[ \cos(A + B + C) = \frac{a - bc}{\sqrt{1 - b^2\sqrt{1 - c^2}}} \frac{b - ca}{\sqrt{1 - c^2\sqrt{1 - a^2}}} \frac{c - ab}{\sqrt{1 - a^2\sqrt{1 - b^2}}} \]
\[ = \frac{(a - bc)(b - ca)(c - ab) - (a - bc + b - ca + c - ab)}{(1 - a^2)(1 - b^2)(1 - c^2)} \cdot 4n^2 \]

³This equals \( 4n^2 = \sin s \sin(s - a) \sin(s - b) \sin(s - c) \) with \( s = (a + b + c)/2 \).
⁴Jean Paul De Gua de Malves (1713-1785), French mathematician.
⁵m.m.: mutatis mutandis, Medieval Latin, “things being changed that have to be changed”
Algebra  Note \( \cos E = \cos(A + B + C - \pi) = -\cos(A + B + C) \).

Euler

\[
2 \cos^2 \frac{1}{2} E = 1 + \cos E = 1 - \cos(A + B + C) = \frac{\text{numerator}_1}{(1 - a^2)(1 - b^2)(1 - c^2)}
\]

with

\[
\text{numerator}_1 = (1 - a^2)(1 - b^2)(1 - c^2) - (a - bc)(b - ca)(c - ab) + (a - bc + b - ca + c - ab)(1 - a^2 - b^2 - c^2 + 2abc)
\]

\[
= (1 - a)(1 - b)(1 - c)(1 + a + b + c)^2 .
\]

Cancel \((1 - a)(1 - b)(1 - c)\) to obtain

\[
2 \cos^2 \frac{1}{2} E = \frac{(1 + a + b + c)^2}{(1 + a)(1 + b)(1 + c)} ,
\]

and

\[
\cos \frac{1}{2} E = \frac{1 + a + b + c}{\sqrt{2(1 + a)(1 + b)(1 + c)}} .
\]

Replacing \(a\) by \(\cos a\), ... m.m. ..., we have

\[
\cos \frac{1}{2} E = \frac{1 + \cos a + \cos b + \cos c}{\sqrt{2(1 + \cos a)(1 + \cos b)(1 + \cos c)}} .
\]

Using \(1 + \cos a = 2 \cos^2 \frac{a}{2}, \) ... m.m. ..., gives the formula by Euler.

Cagnoli

\[
2 \sin^2 \frac{1}{2} E = 1 - \cos E = 1 + \cos(A + B + C) = \frac{\text{numerator}_2}{(1 - a^2)(1 - b^2)(1 - c^2)}
\]

with

\[
\text{numerator}_2 = (1 - a^2)(1 - b^2)(1 - c^2) + (a - bc)(b - ca)(c - ab) - (a - bc + b - ca + c - ab)(1 - a^2 - b^2 - c^2 + 2abc)
\]

\[
= (1 - a)(1 - b)(1 - c)(1 - a^2 - b^2 - c^2 + 2abc) .
\]

Cancel \((1 - a)(1 - b)(1 - c)\) to obtain

\[
2 \sin^2 \frac{1}{2} E = \frac{4n^2}{(1 + a)(1 + b)(1 + c)} ,
\]

and

\[
\sin \frac{1}{2} E = \frac{2n}{\sqrt{2(1 + a)(1 + b)(1 + c)}} .
\]

Replacing \(a\) by \(\cos a\), ... m.m. ..., we have

\[
\sin \frac{1}{2} E = \frac{2n}{\sqrt{2(1 + \cos a)(1 + \cos b)(1 + \cos c)}} .
\]
with
\[2n = \sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}\]

Using \(1 + \cos a = 2 \cos^2 \frac{a}{2}\), ... m.m. ..., gives the formula by Cagnoli.

De Gua This is \(\tan \frac{1}{2}E = \frac{\sin \frac{1}{2}E}{\cos \frac{1}{2}E}\).

Note The formulas of Euler and Cagnoli are related by
\[
(1 + a + b + c)^2 + (1 - a^2 - b^2 - c^2 + 2abc) = 2(1 + a)(1 + b)(1 + c).
\]

**Linear algebra** Let \(A, B, C\) be three unit column vectors. These describe a triangle on the unit sphere in Euclidean space. The sides of the triangle are \(B \cdot C = \mathbf{BC} = \cos a, \ldots, \mathbf{m.m.}\).

For the three-by-three matrix \([ABC]\), we have
\[
\mathbf{t}[ABC][ABC] = \begin{bmatrix}
1 & \cos c & \cos b \\
\cos c & 1 & \cos a \\
\cos b & \cos a & 1
\end{bmatrix}
\]
with determinant
\[
1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c = 4n^2.
\]

Hence, \(\det[ABC] = 2n\).

Using \(1 + B \cdot C = 2 \cos^2 \frac{a}{2}, \ldots, \mathbf{m.m.}\), we obtain
\[
\cos \frac{1}{2}E = \frac{1 + B \cdot C + C \cdot A + A \cdot B}{\sqrt{2(1 + B \cdot C)(1 + C \cdot A)(1 + A \cdot B)}},
\]
\[
\sin \frac{1}{2}E = \frac{\det[ABC]}{\sqrt{2(1 + B \cdot C)(1 + C \cdot A)(1 + A \cdot B)}},
\]
\[
\tan \frac{1}{2}E = \frac{\det[ABC]}{1 + B \cdot C + C \cdot A + A \cdot B}.
\]
These are useful for computation. The last formula contains no radicals.

**Reference**


§127:  
p. 97 Girard, Invention nouvelle en Algèbre, 1629.

§138:  
p. 99 Cagnoli, Traité de Trigonométrie, 1786. [?]  
p.103 Euler, Acta Petropolitana, 1778.  