Notes on a simple Expanding Universe

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1 Introduction

These are notes of calculations to understand signals and horizons in an expanding universe. This involves only the geometrical side of General Relativity applied to the FLRW\textsuperscript{1} Metric in dimension two (one time and one space). Note that this is geometry, not physics.

There are three pages of figures: Signals, Horizons, and Problem. These are the last three pages of these notes.

2 The Metric

The FLRW Metric appears in several forms. The \textit{conventional form}, the most frequently used, is

\[ ds^2 = -c^2 dt^2 + a(t)^2 \left( \frac{1}{1 - kr^2} dr^2 + r^2 d\phi^2 + r^2 \sin^2 \phi d\theta^2 \right) , \]

but we immediately turn to the \textit{geometric form} which is easiest to understand:

\[ ds^2 = -c^2 dt^2 + a(t)^2 \left( d\rho^2 + S(k; \rho)^2 d\phi^2 + S(k; \rho)^2 \sin^2 \phi d\theta^2 \right) , \]

\textsuperscript{1}\text{Friedmann, Lemaître, Robertson, Walker}
by substituting $r = S(k; \rho)$, where $S(k; \rho)$ is $\sin \rho$, $\rho$, or $\sinh \rho$, according as $k = 1, 0, \text{or} -1$, respectively.

Here

- $c$: the speed of light, $m/s$ (meters/second)
- $t$: cosmic time, $s$ (seconds)
- $k$: “curvature”, unitless
- $\theta, \varphi$: two angels for “spherical coördinates, $r$ (radians)
- $\rho$: third spherical coördinate, $r$ (radians)
- $a(t)$: scale factor for expansion of distances, $m/r$ (meters/radian).

In case $k = 1$, the spatial part if the space-time is a three-sphere of radius $a(t)$, and the use of radians is most appropriate.

We will be concerned only with dimension two (one time, one space). The geometric form of the metric is then

$$ds^2 = -c^2 dt^2 + a(t)^2 d\rho^2 ,$$

where space is of one Euclidean dimension and has coördinate $\rho$.

Standing assumption: $a(t)$ is $> 0$ and is differentiable.

Looking ahead, we will introduce

- $\rho$: comoving distance, $r$ (radians)
- $x = a(t)\rho$: proper distance, $m$ (meters)

Cosmic time $t = 0$ is the present, the past and future are $t$ negative and $t$ positive, respectively. The observer is at $\rho = 0$, that is, at $x = 0$. The speed of light $c > 0$ is positive for photons leaving the observer, and negative $c < 0$, same magnitude, for photons approaching the observer. Square roots will always be non-negative.

Units. It is useful to use units to check formulas. Occasionally we follow a formula with its units. For example, the units for the metric $(d2)$ will be written

$$m^2 = -\left(\frac{m}{s}\right)^2 s^2 + \left(\frac{m}{r}\right)^2 r^2 .$$

The usual signs: $+$, $-$, and $=$, are merely for the purpose of locating the terms to which the units refer.
3 Geodesics

3.1 Connection

To obtain curvature and geodesics, one needs the Christoffel symbols of the connection. Those of the *first kind* are obtained from the metric \( g_{ij} \) by:

\[
[ij, k] = \frac{1}{2} \left( \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right),
\]

symmetric in \( i \) and \( j \). Christoffel symbols of the *second kind* are:

\[
\Gamma^k_{ij} = \sum_\kappa g^{\kappa\kappa} [ij, \kappa],
\]

where \([g^{ij}] = [g_{ij}]^{-1}\), matrix inverse, also symmetric in \( i \) and \( j \).

These formulas are to be found in any treatise on Riemannian geometry, or on General Relativity.

3.2 Equations for geodesics

A curve \( \alpha \) is a *geodesic* if its tangent is parallel along the curve. This is the analog of unaccelerated motion in Euclidean space. The coordinates \( x^k = \alpha^k(\lambda) \) of the geodesic \( \alpha \) satisfy the system of second-order equations

\[
\frac{d^2 x^k}{d\lambda^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx^i}{d\lambda} \frac{dx^j}{d\lambda} = 0,
\]

for all \( k \). Note that the Christoffel symbols can contain all of the dependant variables: \( \Gamma^k_{ij}(\ldots, x^h, \ldots) \).

3.3 Dimension two

In space-time of dimension two a geodesic \( \alpha \) is given by parameters: \( t = \alpha^0(\lambda) \) and \( \rho = \alpha^1(\lambda) \). The two equations for this geodesic are

\[
\frac{d^2 t}{d\lambda^2} + \Gamma^0_{00} \left( \frac{dt}{d\lambda} \right)^2 + 2 \Gamma^0_{01} \frac{dt}{d\lambda} \frac{d\rho}{d\lambda} + \Gamma^0_{11} \left( \frac{d\rho}{d\lambda} \right)^2 = 0 \quad (1)
\]
and
\[ \frac{d^2 \rho}{d\lambda^2} + \Gamma^1_{00} \left( \frac{dt}{d\lambda} \right)^2 + 2 \Gamma^1_{01} \frac{dt}{d\lambda} \frac{d\rho}{d\lambda} + \Gamma^1_{11} \left( \frac{d\rho}{d\lambda} \right)^2 = 0 \] (2)

When \( \frac{dt}{d\lambda} \neq 0 \) one can eliminate the parameter \( \lambda \) and obtain a differential equation for \( \rho \) in terms of \( t \). The result \( ^3 \) is
\[ \frac{d^2 \rho}{dt^2} - \Gamma^0_{11} \left( \frac{d\rho}{dt} \right)^3 + \left( \Gamma^1_{11} - 2 \Gamma^0_{01} \right) \left( \frac{d\rho}{dt} \right)^2 + \left( 2 \Gamma^1_{01} - \Gamma^0_{00} \right) \frac{d\rho}{dt} + \Gamma^1_{00} = 0 \ . \]

This is an important formula for the geometry of surfaces.

4 Hubble

4.1 General

The two dimensional metric \( (d2) \): \( ds^2 = -c^2 t^2 + a(t)^2 d\rho^2 \) has coordinates \( x^0 = t \) and \( x^1 = \rho \). The only non-zero Christoffel symbols are:
\[ \Gamma^0_{11} = \frac{a \dot{a}}{c^2} , \quad \Gamma^1_{01} = \frac{\dot{a}}{a} , \quad \text{and} \quad \Gamma^1_{10} = \frac{\dot{a}}{a} , \]
where \( \dot{a} = da/dt \).

The equation for geodesics becomes
\[ \frac{d^2 \rho}{dt^2} - \frac{a \dot{a}}{c^2} \left( \frac{d\rho}{dt} \right)^3 + 2 \frac{\dot{a}}{a} \frac{d\rho}{dt} = 0 \ , \quad \text{(h)} \]
with units
\[ \frac{r}{s^2} - (r^{-2} s) \left( \frac{r}{s} \right)^3 + (s^{-1}) \frac{r}{s} = 0 \ . \quad \text{units (h)} \]

Note that there are no restrictions: a geodesic can be time-like, null, or space-like \( (t \neq \text{constant}) \).

\(^2\)Into (2) substitute \( \frac{d\rho}{dx} = \frac{d\rho}{dt} \frac{dt}{dx} \) and \( \frac{d^2 \rho}{dx^2} = \frac{d^2 \rho}{dt^2} \left( \frac{dt}{dx} \right)^2 + \frac{d\rho}{dt} \frac{d^2 t}{dx^2} \), multiply (1) by \( \frac{d\rho}{dt} \), subtract from (2), and divide by \( (\frac{dt}{dx})^2 \).

One easily checks that the integrals
\[ \rho(t) = \rho(t_0) + \int_{t_0}^{t} \frac{c}{a(t')} \, dt' \]
(c positive or negative) are solutions. These solutions are null geodesics.\(^4\) A choice of \(c\) and a choice of \(\rho(t_0)\) at \(t = t_0\) are the two initial conditions for this second order ordinary differential equation. Initial conditions for non-null geodesics will be addressed later.

The time derivative of the logarithm of the expansion factor is known as the Hubble constant \(H\). It need not be constant:
\[ \frac{\dot{a}(t)}{a(t)} = H(t) \quad \text{and} \quad a(t) = a(t_0) \exp \int_{t_0}^{t} H(t') \, dt' . \]
The units of \(H(t)\) are \((m/rs)/(m/r) = s^{-1}\).

The scale factor \(a(t)\) relates the comoving distance \(\rho\) and the proper distance \(x\) at the same cosmic time \(t\) by \(x = a(t) \rho\).

Now, \(\dot{a}(t) = H(t) a(t)\) and for a null geodesic \(\dot{\rho}(t) = c/a(t)\). From \(x = a \rho\) we have \(\dot{x} = \dot{a} \rho + a \dot{\rho} = Ha \rho + a(c^2)\), so
\[ \dot{x} = H(t) x(t) + c . \]
This is interpreted as
\[ \begin{pmatrix} \text{speed of photon} \\
\text{along null geodesic} \end{pmatrix} = \begin{pmatrix} \text{expansion} \\
\text{of universe} \end{pmatrix} + \begin{pmatrix} \text{speed} \\
\text{of light} \end{pmatrix} \]
From the viewpoint of the comoving distance \(\rho\), light is slowing down: \(\dot{\rho}(t) = c/a(t)\).

### 4.2 Quadrature

We return to the equation for a geodesic
\[ \frac{d^2 \rho}{dt^2} - \frac{a \ddot{a}}{c^2} \left( \frac{d \rho}{dt} \right)^3 + 2 \frac{\dot{a}}{a} \frac{d \rho}{dt} = 0 , \quad (h) \]
\(^4\)For \(\dot{\rho} = c/a(t)\), and from \((d2)\) we have \(\left(\frac{d \rho}{dt}\right)^2 = c^2 + a(t)^2 \dot{\rho}^2\). Hence, \((ds/dt)^2 = 0\).
where $\dot{a} = da/dt$. And, $a(t)$ is known. There are no restrictions: a geodesic can be time-like, null, or space-like ($t \neq$ constant).

It is remarkable that this equation is solvable by quadrature – that is, the solution is expressable as an integral, but the integral may or may not be possible to write it in terms of well-known functions.

Set $u(t) = \dot{\rho}(t)/c$. The equation becomes

$$\frac{du}{dt} - a \dot{a} u^3 + 2 \frac{\dot{a}}{a} u = 0$$

Multiplying by $a(t)^2$ we have

$$\frac{d}{dt} (a^2 u) - a^3 \dot{a} u^3 = 0 \quad \text{and} \quad \frac{d}{dt} (a^2 u) = (a^2 u)^3 \frac{1}{a^3} \frac{da}{dt}.$$

Write this last equation as

$$\frac{d(a^2 u)}{(a^2 u)^3} = \frac{da}{a^3}.$$

We integrate formally, and check later that the solution is valid. (It may be that $a$ cannot be an independent variable.) We have:

$$-\frac{1}{2} \frac{1}{(a^2 u)^2} = -\frac{1}{2} \frac{1}{a^2} + constant.$$

Denote the constant of integration by $constant = \frac{1}{2} B$. Then

$$B = \frac{1}{a^2} - \frac{1}{a^4 u^2}. \quad (0)$$

Recalling that $u = \dot{\rho}/c$, (0) becomes

$$B = \frac{1}{a(t)^2} \left(1 - \frac{c^2}{a(t)^2 \dot{\rho}(t)^2}\right), \quad (1)$$

which is a constant; and solving (0) for $u$ gives

$$\dot{\rho}(t) = \frac{c}{a(t)} \frac{1}{\sqrt{1 - Ba(t)^2}}. \quad (2)$$

(Keep in mind that $c$ may be positive or negative.)
Equation (1) evaluated at \( t = t_0 \) shows that \( \dot{\rho}(t_0) \) determines the constant \( B \). Equation (2) evaluated at \( t = t_0 \) shows that \( B \) determines \( \dot{\rho}(t_0) \).

And, one may now check, by substitution, that (2) is indeed a solution to the differential equation (h).

Note that the constant \( B \) has units \( m^{-2}r^2 \), and \( Ba^2 \) is unitless.

From \( d\rho = \dot{\rho} dt \), equation (2), and the metric (d2), we have

\[
\left( \frac{ds}{dt} \right)^2 = c^2 \frac{Ba^2}{1 - Ba^2} .
\]

So, the kind of geodesic is determined\(^5\) by \( B \):

<table>
<thead>
<tr>
<th>geodesic</th>
<th>((ds/dt)^2)</th>
<th>(Ba^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>timelike</td>
<td>(-c^2)</td>
<td>(Ba^2 = \pm \infty)</td>
</tr>
<tr>
<td>null</td>
<td>(-c^2 &lt; (\cdot)^2 &lt; 0)</td>
<td>(-\infty &lt; Ba^2 &lt; 0)</td>
</tr>
<tr>
<td>spacelike</td>
<td>(= 0)</td>
<td>(Ba^2 = 0)</td>
</tr>
<tr>
<td>none</td>
<td>(&gt; 0)</td>
<td>(0 &lt; Ba^2 &lt; 1)</td>
</tr>
</tbody>
</table>

\(5\)It is worth sketching \( y = x/(1 - x) \) for this table.

Note:
- The radicands are positive for \(-\infty < Ba^2 < 1\).
- For \(Ba^2 = \pm \infty\), \( \rho(t) \) is constant.

Aside. One also has \( 1 - Ba^2 = \frac{c^2}{a^2 \dot{\rho}(t_0)^2} \) with which one can compare the speed of light \( a \dot{\rho}(t_0) \), as viewed in the expanding comoving coordinates, with \( c \).

Finally, \( \rho(t) \) is determined by

\[
\rho(t) = \rho(t_0) + \int_{t_0}^{t} \frac{c}{a(t')} \frac{1}{\sqrt{1 - Ba(t')^2}} dt' .
\]

Earlier we had observed that the integrals

\[
\rho(t) = \rho(t_0) + \int_{t_0}^{t} \frac{c}{a(t')} dt'
\]

are solutions to (h) – null geodesics. These are the special case \( B = 0 \).
4.3 Exponential

To obtain a non-trivial example in which all calculations may be carried out explicitly, we assume that the Hubble constant is indeed a constant $H$:

$$a(t) = a(0) e^{Ht}.$$ 

Null geodesics: $Ba^2 = 0$, the paths of photons.

Here we will use $a(t) = a(z) e^{H(t-z)}$ with $t = z$ for the present – allowing the possibility of choosing a different time for the present. This will also affirm that the presence of $z$ will not affect the exponential formulas, and that we may later use $z = 0$.

From $\dot{a}(t) = c/a(t),$

$$\dot{a}(t) = \frac{c}{a(z)e^{H(t-z)}} = \frac{c}{a(z)e^{-Hz}} \frac{1}{e^{Ht}}.$$

Then $\rho(t)$ is determined by

$$\rho(t) - \rho(t_0) = \frac{c}{a(z)e^{-Hz}} \int_{t_0}^{t} \frac{1}{e^{Ht'}} dt' = \frac{c}{a(z)e^{-Hz}} \frac{1}{H} (e^{-Ht} - e^{-Ht_0})$$

$$= \frac{c}{a(z)e^{-Hz}} \frac{1}{e^{Ht_0}} \frac{1}{H} (1 - e^{-H(t-t_0)}) = \frac{c}{a(t_0)} \frac{1 - e^{-H(t-t_0)}}{H}.$$

And, to determine $x(t)$:

$$x(t) = a(t) \rho(t), \quad a(t) = a(t_0) e^{H(t-t_0)}, \quad \rho(t) = \rho(t_0) + \frac{c}{a(t_0)} \frac{1 - e^{-H(t-t_0)}}{H}$$

yields

$$x(t) = x(t_0) e^{H(T-t_0)} + c \frac{e^{H(t-t_0)} - 1}{H}.$$

Aside. For $H(t-t_0)$ small, we can expand and obtain

$$\rho(t) - \rho(t_0) = \frac{c}{a(t_0)} \left( (t-t_0) - \frac{1}{2} H(t-t_0)^2 + \frac{1}{6} H^2(t-t_0)^3 - \ldots \right),$$

and then $\rho(t) - \rho(t_0) \approx \frac{c}{a(t_0)} (t-t_0).$
Timelike geodesics: $-\infty < Ba^2 < 0$, the paths of massive particles. Now we use $a(t) = a(0) \, e^{Ht}$.

From
\[
\dot{\rho}(t) = \frac{c}{a(t)} \frac{1}{\sqrt{1 - Ba(t)^2}} .
\]
we have
\[
\dot{\rho}(t) = \frac{c}{a(0)e^{Ht}} \frac{1}{\sqrt{1 - B(a(0)e^{Ht})^2}} = \frac{c}{a(0)} \frac{1}{e^{Ht}} \frac{1}{\sqrt{1 - \beta e^{2Ht}}} ,
\]
where
\[
\beta = Ba(0)^2 .
\]
Then $\rho(t)$ is determined by
\[
\rho(t) - \rho(t_0) = \frac{c}{a(0)} \int_{t_0}^{t} \frac{1}{e^{Ht'}} \frac{1}{\sqrt{1 - \beta e^{2Ht'}}} \, dt' = \frac{c}{a(0)} \frac{1}{H} \left[ - \frac{1}{e^x} \sqrt{1 - \beta e^{2x}} \right]_{x=Ht}^{x=Ht_0}
\]
\[
= \frac{c}{a(0)} \sqrt{1 - \beta e^{2Ht_0}} \frac{1}{e^{Ht_0}} \frac{1}{H} \left( 1 - e^{-H(t-t_0)} \frac{\sqrt{1 - \beta e^{2Ht}}}{{\sqrt{1 - \beta e^{2Ht_0}}}} \right)
\]
\[
= \frac{c}{a(t_0)} \sqrt{1 - \beta e^{2Ht_0}} \frac{1 - e^{-H(t-t_0)} \sqrt{1 - \beta e^{2Ht}}}{\sqrt{1 - \beta e^{2Ht_0}}} H .
\]
And, $x(t)$ can be determined in a manner like the above.

Timelike geodesics: $Ba^2 = -\infty$, the paths of stationary particles.

Here $\dot{\rho}(t) = 0$, so $\rho(t) = \text{constant}$, and $x(t)$ is increasing ($H > 0$).

5 Signals

5.1 Example - exponentially expanding universe

For an example of an expanding universe we will use $a(t) = a(0) \, e^{Ht}$, $H > 0$ a constant.
The space-time graphs are drawn with the independent variable $t$ (cosmic time) as the ordinate (vertical), and the dependent variable $\rho$ or $x$ as the abscissa (horizontal). See first Figure at the end of this article.

Comoving distance $\rho$. A signal is sent along a null-geodesic from location $\rho = 0$ at time $t_0$ and arrives at location $\rho = \rho_1$ at time $t_1$. The time $t_1$ is to be determined.

We have

$$\dot{\rho}(t) = \frac{c}{a(t_0) e^{H(t-t_0)}} \quad \text{and} \quad \rho(t) = \frac{c}{a(t_0)} \frac{1 - e^{-H(t-t_0)}}{H}.$$ 

The least distance to which a signal may not arrive is $\rho = c/a(t_0)H$.

The arrival time is

$$t_1 = t_0 + \frac{1}{H} \ln \frac{1}{1 - \frac{\rho_1}{c} e^{H(t-t_0)}} \quad \text{for} \quad a(t_0) \rho_1 < \frac{c}{H}.$$ 

For $\rho_1 \geq c/a(t_0)H$, there is no solution.

To pass from comoving distances to proper distances we use $x = a(t) \rho$. In the case of this expanding universe $a(t) = a(t_0) e^{H(t-t_0)}$ meters/radian.

Proper distance $x$. A signal is sent along a null-geodesic from location $x = 0$ at time $t_0$ and arrives at location $x = x_1$ at time $t_1$. The time $t_1$ is to be determined.

Now

$$x(t) = a(t_0) e^{H(t-t_0)} \times \frac{c}{a(t_0)} \frac{1 - e^{-H(t-t_0)}}{H} = c e^{H(t-t_0)} - 1.$$ 

Then $t_1$ is determined by solving

$$c e^{H(t_1-t_0)} - 1 = x_1 e^{H(t_1-t_0)}.$$ 

The result is

$$t_1 = t_0 + \frac{1}{H} \ln \frac{1}{1 - \frac{x_1}{c} e^{H(t-t_0)}} \quad \text{for} \quad x_1 < \frac{c}{H}.$$ 

For $x_1 \geq c/H$ there are no solutions.

Observe that the formulas for $\rho$ and $x$ are entirely consistant.

If one examines these formulas as $H \to 0$, formulas for the case a static universe will be recognised, especially $x(t) = c(t-t_0)$.

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Useful:

$$\frac{1}{H} \ln \frac{1}{1 - uH} = u + \frac{1}{2} u^2 H + \frac{1}{3} u^3 H^2 + \frac{1}{4} u^4 H^3 + \cdots.$$
5.2 General - expanding universe

For the general case of an expanding (and contracting) universe, we assume only that \( a(t) \) is \( > 0 \).

The space-time graphs will be visualized with the independent variable \( t \) (cosmic time) as the ordinate (vertical), and the dependent variable \( \rho \) or \( x \) as the abscissa (horizontal).

Comoving distance \( \rho \). A signal is sent along a null-geodesic from location \( \rho = 0 \) at time \( t_0 \) and arrives at location \( \rho = \rho_1 \) at time \( t_1 \). The time \( t_1 \) is to be determined.

For null geodesics \( \dot{\rho}(t) = c/a(t) \), so \( d\rho = c \frac{dt}{a(t)} \). Then

\[
\rho(t) = c \int_{t_0}^{t} \frac{dt'}{a(t')}
\]

To determine \( t_1 \), one solves

\[
c \int_{t_0}^{t_1} \frac{dt'}{a(t')} = \rho_1
\]

for \( t_1 \). If \( \rho_1 \geq c \int_{t_0}^{\infty} \frac{dt'}{a(t')} \), as in the example, there is no solution.

To pass from comoving distances to proper distances we use \( x = a(t) \rho \). Proper distance \( x \). A signal is sent along a null geodesic from location \( x = 0 \) at time \( t_0 \) and arrives at location \( x = a(t_1) x_1 \) at time \( t_1 \). The time \( t_1 \) is to be determined.

Now the signal follows the path

\[
x = c a(t) \int_{t_0}^{t} \frac{dt'}{a(t')}
\]

and the receiver is at \( x = a(t) \rho_1 \). Equating these at \( t = t_1 \):

\[
c a(t_1) \int_{t_0}^{t_1} \frac{dt'}{a(t')} = a(t_1) \rho_1
\]

Cancel \( a(t_1) \) to obtain the same formula as before to be solved for \( t_1 \):

\[
c \int_{t_0}^{t_1} \frac{dt'}{a(t')} = \rho_1
\]
5.3 Horizons

It is common to express horizons in terms of comoving coördinates; we do that here.

An event within the particle horizon has been observed in the past or is observed at the present. An event beyond the event horizon cannot be observed at any time - including the future. See the second Figure at the end of this article.

Again the space-time graph are drawn with the independent variable $t$ (cosmic time) as the ordinate (vertical), and the dependent variable $\rho$ as the abscissa (horizontal).

On the ordinate:
- Big Bang: $t = t_0$ which is negative.
- The present: $t = 0$.

A photon travels along a null geodesic from an event at $(t_0, \rho_0)$ and is received at $(t_1, 0)$.

On the abscissa:
- Observer: $\rho = 0$.
- Particle horizon: $\rho_{PH}$
- Event horizon: $\rho_{EH}$

We take the speed of light to be negative and write $\dot{\rho}(t) = -c/a(t)$ with $c > 0$. Time $t_1$ and comoving distance $\rho_0$ are related by

$$0 - \rho_0 = \int_{t_0}^{t_1} -\frac{c}{a(t')} dt' .$$

One thinks in terms of solving for $t_1$ given $\rho_0$.

Especially note the particle horizon: $t_1 = 0$:

$$\rho_{PH} = \int_{-|t_0|}^{0} \frac{c}{a(t')} dt' ,$$

and the event horizon $t_1 = \infty$:

$$\rho_{EH} = \int_{-|t_0|}^{\infty} \frac{c}{a(t')} dt' ,$$

where $-|t_0|$ emphasizes that $t_0$ is negative.
<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$t_1$</th>
<th>seen</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; \rho &lt; \rho_{PH}$</td>
<td>$t_0 &lt; t_1 &lt; 0$</td>
<td>past</td>
</tr>
<tr>
<td>$\rho = \rho_{PH}$</td>
<td>$t_1 = 0$</td>
<td>present</td>
</tr>
<tr>
<td>$\rho_{PH} &lt; \rho &lt; \rho_{EH}$</td>
<td>$0 &lt; t_1$</td>
<td>future</td>
</tr>
<tr>
<td>$\rho_{EH} \leq \rho$</td>
<td>no solution</td>
<td>never</td>
</tr>
</tbody>
</table>

Example: The familiar $a(t) = a(0) e^{Ht}$, $H > 0$.

We have

$$\rho_{PH} = \int_{t_0}^{0} \frac{c}{a(0) e^{Ht'}} dt' = \frac{c}{a(t_0)} \frac{1}{H} (1 - e^{Ht_0})$$

and

$$\rho_{EH} = \int_{t_0}^{\infty} \frac{c}{a(0) e^{Ht'}} dt' = \frac{c}{a(t_0)} \frac{1}{H},$$

where $Ht_0 < 0$.

We have seen latter in the previous sections: $x = a(t_0) \rho_{EH} = c/H$ determines the distances $\rho$ and $x$ past which solutions for $t_1$ do not exist. It is commonly called the "cosmological horizon distance".

**Problem** for future thought.

The ship is at $O$, and a distant galaxy at $G$. The ship signals from $O$ to $G$ that it is coming for a visit. The signal reaches $G$ and the ship, fairly fast, also reaches $G$. The ship signals back from $G$ to $O$ that it is returning. The universe is expanding. The signal reaches $O$, but the ship cannot. A space-time diagram is easy to draw that will convince you that this can happen. The formulas regarding timelike geodisics $-\infty < Ba^2 < 0$ are available. The problem is to determine parameters for this and carry out the necessary calculations. See third Figure at the end of this article.

**Reference**

Tamara M. Davis, Charles H. Lineweaver

Expanding Confusion: common misconceptions of cosmological horizons and the superluminal expansion of the Universe

13 Nov 2003 version v2


SIGNALS - EXPANDING UNIVERSE $a(t) = a(0)e^{Ht}$, $H > 0$ constant

COMING DISTANCE

\[ d(t) = \frac{c}{a(t)} \frac{1 - e^{-H(t-x_0)}}{H} \]

\[ d(t) = \frac{c}{a(t_0)H} \left( 1 - e^{-\frac{H(t-x_0)}{c}} \right) \]

\[ x = c \frac{H(t-x_0)}{c} \]

PROPER DISTANCE

\[ d(t) = \frac{c}{a(t)} \frac{H(t-x_0)}{H} \]

\[ d(t) = \frac{c H(t-x_0)}{H} - 1 = x + H(t-x_0) \]

\[ \frac{c H(t-x_0)}{H} - 1 = x + H(t-x_0) \]

\[ \frac{c H(t-x_0)}{H} - 1 = x_1 + H(t-x_0) \]

\[ x_1 = \frac{c}{H} H(t-x_0) \leq (t - x_0) \]

FOR $x_1 < \frac{c}{H}$

\[ r_1 = x_0 + \frac{1}{H} \ln \frac{1}{1 - \frac{c}{a(t_0)H}} \]

\[ r_1 < \frac{c}{a(t_0)H} \]

For $r_1 < \frac{c}{a(t_0)H}$

\[ x = \frac{c}{H} e^{H(t-x_0)} \]

\[ \frac{c H(t-x_0)}{H} - 1 = x_1 + H(t-x_0) \]

\[ \frac{c H(t-x_0)}{H} - 1 = x_1 + H(t-x_0) \]

\[ x = \frac{c}{H} e^{H(t-x_0)} \leq (t - x_0) \]

\[ For x_1 \geq \frac{c}{H} \]

\[ No Solution \]

\[ x = \frac{c}{H} e^{H(t-x_0)} \leq (t - x_0) \]
WORLD LINE OF OBSERVER

WORLD LINE OF GALAXY

LHIGHT TRAVEL

K2 \( x = \alpha(t) x_0 \)

VIEW FROM OBSERVER AT C