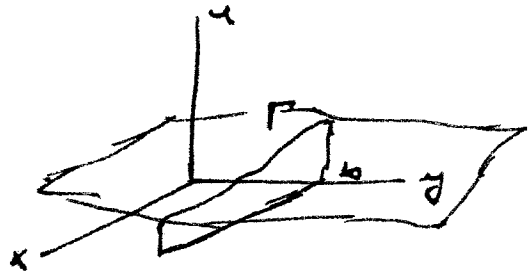
② For the equation $u_y = u_x^3$,
a) find the solution satisfying $u(x,0) = 2x^{3/2}$;
b) show that every solution differentiable for all x, y is linear.

~~Ex.~~ F. John 1.30 No. 2.

Solu'n

a) Γ' { $x = s$; $y = b$ const } Γ; $u = f(s)$; $p = f'(s)$; $q = f'(s)^3$



Into Γ:
$$\begin{cases} u_0' = p x_b' + q y_b' \\ F(x_0, y_0, u_0, p_0, q) = 0 \end{cases} \rightarrow \begin{matrix} f'(s) = p \cdot 1 + q \cdot 0 \\ q - p^3 = 0 \end{matrix}$$

solu'n: $p_0(s) = f'(s)$, $q_0(s) = f'(s)^3$.

$F(x, y, u, p, q) = q - p^3$

$$\text{CE} \begin{cases} \frac{dx}{dt} = -3p^2 \\ \frac{dy}{dt} = 1 \\ \frac{du}{dt} = p(-3p^2) + q(1) = -3p^3 + q \overset{F=0}{=} -2p^3 \text{ or } -2q \\ \frac{dp}{dt} = 0 \\ \frac{dq}{dt} = 0 \end{cases}$$

solu'n:
$$\left.\begin{cases} x = s - 3f'(s)^2 \tau \\ y = b + \tau \\ u = f(s) - 2f'(s)^3 \tau \\ p = f'(s) \\ q = f'(s)^3 \end{cases}\right\} \rightarrow u = u(x, y)$$

Case given: $f(s) = 2s^{3/2}$, $b = 0$. $f'(s) = 3s^{1/2}$

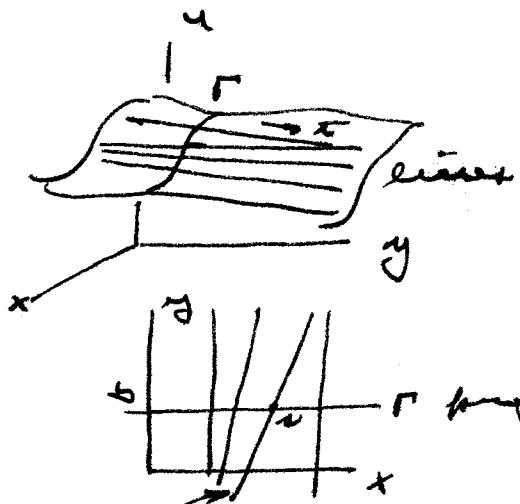
$$\left.\begin{cases} x = s - 27 s \tau \\ y = \tau \\ u = 2s^{3/2} - 54 s^{3/2} \tau \end{cases}\right\} \begin{matrix} s = x(1-27y)^{-1} \\ \tau = y \end{matrix}$$

$$u = 2x^{3/2}(1-27\tau) = 2\left(x(1-27y)^{-1}\right)^{3/2}(1-27y)$$

$$\boxed{u = 2x^{3/2}(1-27y)^{-1/2}}$$

Curve separation of variables.

b) Green regular soln, Γ = solen surface intersect base $y=b$. Characs. from bdy of Γ must not cross (integral curves of field are surface from (range cond.).



$$\begin{cases} x = \xi - 3f'(\xi)^2\tau \\ y = b + \tau \end{cases}$$

$$x = \xi - 3f'(\xi)^2(y-b)$$

$$x + 3f'(\xi)^2 y = \xi + b3f'(\xi)^2$$

For projected characteristics not to cross, need these function for different ξ:

$$\begin{vmatrix} 1 & 3f'(\xi_0)^2 \\ 1 & 3f'(\xi)^2 \end{vmatrix} = 0 \quad \text{for } \xi \neq \xi_0.$$

$\rightarrow f'(\xi) = \text{const} \Rightarrow a \rightarrow f(\xi) = a\xi + \beta c'.$

$$\begin{cases} x = \xi - 3a^2\tau \\ y = b + \tau \\ u = (a\xi + c) - 2a^3\tau \end{cases} \rightarrow \quad \xi = x + 3a^2y - 3a^2b$$
$$\tau = y - b$$

$$u = a(x + 3a^2y - 3a^2b) + c - 2a^3(y-b)$$

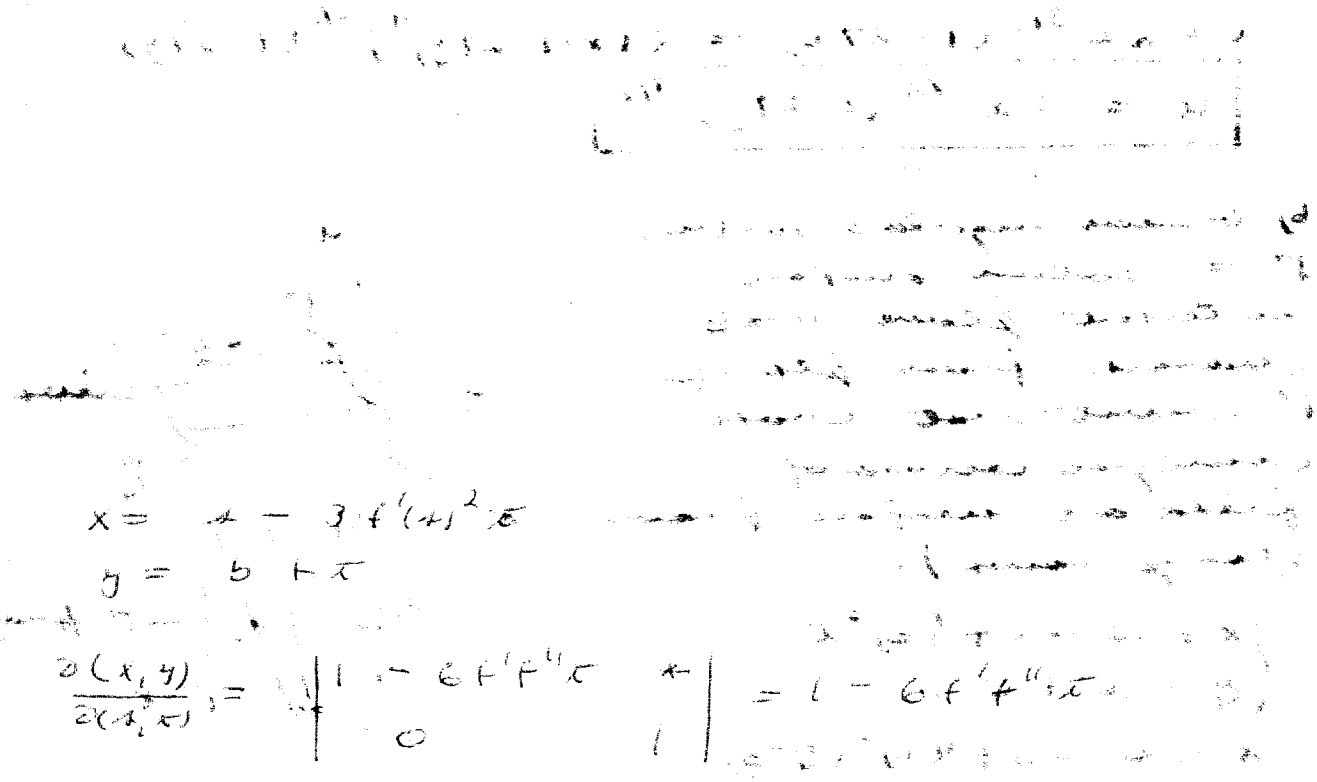
$$\boxed{u = ax + a^3y + (\text{const})}$$

Every soln is envelope of plane solns.

Aside. For y fixed. & arbitrary $f(\xi)$!

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = f'(1-6f'f''\tau)/(1-6f'f''\tau) = f'(\xi).$$

$$\frac{\partial y}{\partial \xi} = f' - 6f'^2f''\tau, \quad \frac{\partial x}{\partial \xi} = 1 - 6f'f''\tau$$
$$= f'(1 - 6f'f''\tau)$$



Alternate solutions

$u = \varphi(x,y)$ every where differentiable solution.

$$\Pi' \begin{cases} x = s \\ y = b \\ u = \varphi(s,b) \\ p = \varphi_x(s,b) \\ q = \varphi_x(s,b)^3 \end{cases} \quad \text{Solve} \quad \varphi \in E_2 \quad \begin{cases} x = s - 3\varphi_x(s,b)^2 t \\ y = b + t \\ u = \varphi(s,b) - 2\varphi_x(s,b)^3 t \\ p = \varphi_x(s,b) \\ q = \varphi_x(s,b)^3 \end{cases} \Big\} \cdot C_s.$$

$u = \varphi(x,y)$!

$$\varphi(s,b) - 2\varphi_x(s,b)^3 t = \varphi(s - 3\varphi_x(s,b)^2 t,\ b + t)$$

$\frac{\partial}{\partial s}$! ← $\varphi_x(s,b)$ etc

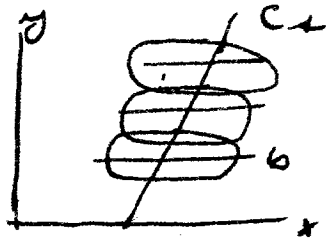
$$\varphi_x - 6\varphi_x^2 \varphi_x t = \varphi_x(s - 3\varphi_x^2 t,\ b + t)(1 - 6\varphi_x \varphi_{xx} t)$$

t small $\longrightarrow$ $1 - 6\varphi_x \varphi_{xx} t \neq 0 \longrightarrow$

$$\varphi_x(s,b) = \varphi_x(s - 3\varphi_x(s,b)^2 t,\ b + t)$$

C_s ← projected characteristic.

Thus: φ_x constant along C_s for small t.



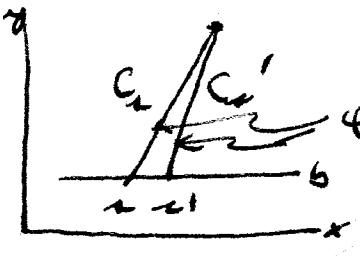
→ φ_x constant along entire C_s.

C_s : line $x + 3\varphi_x(s,b)^2 y = s + 3\varphi_x(s,b)^2 b$.

Suppose $\varphi_x(s,b)$ not constant in s ;

$\varphi_x(s,b) \neq \varphi_x(s',b)$

s, s' nearby



φ_x constant on C_s & $C_{s'}$ } → Contradiction.
C_s & $C_{s'}$ not parallel.

Thus: $\varphi_x(x,y)$ a constant for all x,y.

$\varphi_y = \varphi_x^3$ → φ_y ——— " ———.