① Assume that the rate of deposit or removal of sand on the bed of a stream is proportional to the derivative of the velocity of the flow of water with respect to distance in the direction of flow.
a) Show that the height $u(x,t)$ of the sand, above an arbitrary zero level, in a stream bed, above an arbitrary zero level, satisfies the PDE

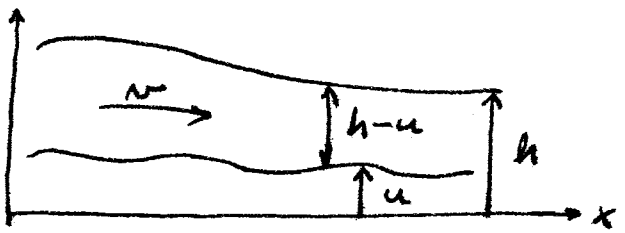
$$(h-u)^2 \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 ,$$

where $h(x,t)$ is the height of the water surface, x is distance in the direction of flow, and a is a constant.
b) Assume h is constant and find the solution (implicit) of this PDE which satisfies the initial condition $u(x,0) = f(x)$.

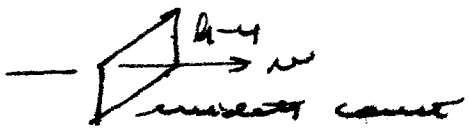
Soln

See Smollen p. 84 No. 1.

a)



v = flow rate.



$$v \propto \frac{1}{h-u} , \qquad v = \frac{1}{a} \frac{x}{h-u} \quad (a = \text{const})$$

$$\frac{\partial u}{\partial t} \propto \frac{\partial v}{\partial x} \propto \frac{1}{(h-u)^2} \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} = \pm a \frac{1}{(h-u)^2} \frac{\partial u}{\partial x} , \qquad a = \text{const.}$$

$$\boxed{(h-u)^2 \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0}$$

$h = h(x,t)$
RHS $= a \frac{\partial h}{\partial x}$

b) PDE is quasi-linear. $A = a$, $B = (h-u)^2$, $C = 0$

$$\Gamma \begin{cases} x = x_0(s) = s \\ t = t_0(s) = 0 \\ u = u_0(s) = f(s) \end{cases}$$

$$CE\begin{cases}\frac{dx}{d\tau}=a\\ \frac{dt}{d\tau}=(h-u)^2\\ \frac{du}{d\tau}=0\end{cases}\longrightarrow\begin{cases}x=s+a\tau\\ t=(h-f(s))^2\tau\\ u=f(s)\end{cases}$$

$$\begin{cases}x=s+a\tau\\ t=(h-f(s))^2\tau\end{cases}\rightarrow\frac{x-s}{a}=\frac{t}{(h-f(s))^2}\rightarrow s=S(x,t)$$

$$\begin{cases}s=S(x,t)\\ \tau=\frac{x-S(x,t)}{a}\end{cases}$$ Change of variables.

$$u=f(s)\longrightarrow u=f(S(x,t)).$$

Or Eliminate s from $\begin{cases}\frac{x-s}{a}=\frac{t}{(h-f(s))^2}\\ u=f(s)\end{cases}$ to get implicit eqn for u.

$$\frac{x-s}{a}=\frac{t}{(h-u)^2}\longrightarrow s=x-\frac{at}{(h-u)^2}.$$

$$u=f(s)\longrightarrow\boxed{u=f\left(x-\frac{at}{(h-u)^2}\right)}$$

Aside.

$$\Delta(t)=\frac{\partial(x,t)}{\partial(s,\tau)}(s,0)=\frac{\partial x_0}{\partial s}B-\frac{\partial t_0}{\partial s}A$$

$$=1\cdot(h-f(s))^2-0\cdot a=(h-f(s))^2$$

$\Delta\neq 0$ for $u<h$.