Math 132A Winter 1981 Fillmore

Final Due March 19

(1) The height $u(x,t)$, above an arbitrary zero level, of sand deposited by flowing water on the bed of a stream satisfies the PDE

$$(h - u)^2 \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad ,$$

where h is the constant height of the water surface, x is distance in the direction of flow, and a is a constant.

a) Describe the singularities of the solution of this PDE with initial condition $u(x,0) = f(x)$.

b) Especially, examine the case when $f(x)$ is linear on an interval and constant elsewhere.

c) Interpret the results in terms of the "breaking" of "sand waves".

(2) a) Show that the solution of the initial value problem for the nonhomogeneous wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{c^2} \dfrac{\partial^2 u}{\partial t^2} = f(x,t) \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

is given by

$$u(x,t) = -\frac{c}{2} \iint_T f(\xi,\tau)\, d\xi\, d\tau \quad ,$$

where T is the triangle bounded by the x-axis and the two characteristics through the point (x,t). Hint: Change coordinates so that the characteristics are coordinate axes.

b) Solve

$$\begin{cases} u_{xx} - u_{tt} = 1 \quad (t > 0 \text{ , all } x) \\ u(x,0) = x \\ u_t(x,0) = 0 \end{cases}$$

without using Fourier transforms. Hint: Superposition of solutions.

(3) From Gustafson PDEs, Problem 1.9.9, Confirmation Exercises (p.110): Show that either the backward heat equation (Prob. 2(a)) or the Cauchy problem for the Laplace equation (Prob. 2(b)) is not well-posed due to lack of initial data stability.