The wave equation - continued.

In several dimensions

PDE $\boxed{\Delta u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0}$ $\qquad \Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$

IC $\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$ $\qquad$ Γ on $x = (x_1, \ldots, x_n)$ plane

$\frac{\partial u}{\partial n}$ is $u_t(x,0)$

Characteristics

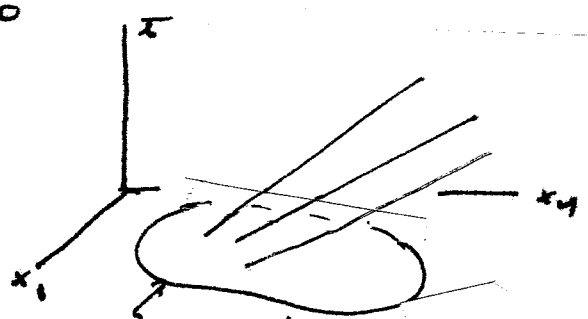
PDE for charac. hypersurfaces: $\sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2 - \frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 = 0$.

CEs for bicharac.

$$\begin{cases} \frac{dx}{d\tau} = b \\ \frac{dt}{d\tau} = -\frac{1}{c^2} q \\ \frac{du}{d\tau} = b^2 - \frac{1}{c^2} q^2 = 0 \\ \frac{db}{d\tau} = 0 \\ \frac{dq}{d\tau} = 0 \end{cases}$$

$F = \frac{1}{2}\left(\sum b_i^2 - \frac{1}{c^2} q^2\right)$

$b = (b_1, \ldots, b_n)$, etc.



Solve:

$$\begin{cases} x = x_0(s) + b_0(s)\,\tau \\ t = -\frac{1}{c^2} q_0(s)\,\tau \\ u = u_0(s) \\ b = b_0(s) \\ q = q_0(s) \end{cases}$$

$$\begin{cases} x = x_0(s) \\ t = t_0(s) = 0 \\ s = (s_1, \ldots, s_n) \end{cases}$$

(Eg. $x = s$).

Eliminate τ:

$$x - x_0(s) = c\left(\frac{-c\,b_0(s)}{q_0(s)}\right) t$$

or

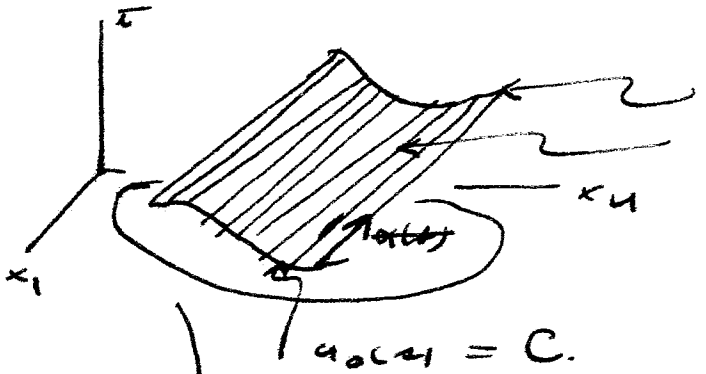
$$x = x_0(s) + \alpha(s)\,c\,t$$

$\alpha(s)$ = unit vectors

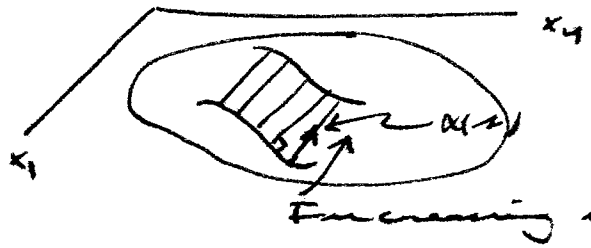
Bicharacteristics.

$$\begin{cases} \alpha = -\frac{c\,b_0}{q_0} \\ \alpha^2 = 1 \end{cases}$$

Characteristic hypersurface: $\begin{cases} \varphi(x,\tau) = C = \text{const.} \\ u = \varphi(x,\tau) \end{cases} \rightarrow u_0(t) = C.$



Level surface $\tau = \text{const.}$ projected to x-space.



Strip condition:

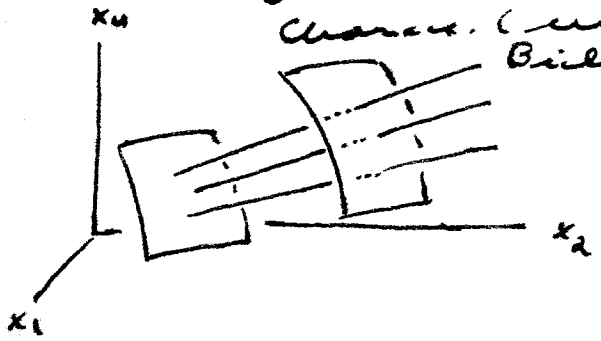
$$\frac{\partial u_0}{\partial t_j}(t) = \sum_{i=1}^{4} b_{0i}(t) \frac{\partial x_{0i}}{\partial t_j}(t) + \tau_0(t) \frac{\partial \tau_0}{\partial t_j}(t) \qquad j = 1, 2, 3, 4.$$

$$\left.\begin{array}{l} \tau_0(t) = 0 \\ u_0(t) = C \end{array}\right\} \rightarrow \sum_{i=1}^{4} b_{0i}(t) \frac{\partial x_{0i}}{\partial t_j}(t) = 0$$

$\frac{\partial x_0}{\partial t_1}, \dots, \frac{\partial x_0}{\partial t_4}$ tangent to $u_0(t) = C$ in x-space.

$\alpha = -\frac{c b_0}{\tau_0} \rightarrow \alpha \perp \frac{\partial x_0}{\partial t_1}, \dots, \frac{\partial x_0}{\partial t_4}$

So: $\alpha(t)$ is perpendicular to characteristic hypersurface through $x = x_0(t) + \alpha(t)\, c\tau$.



Normal congruence of lines.

Spherical means

$f(x)$ scalar fcn $\quad F = \mathcal{M}_r[f]$

$x = (x_1, \dots, x_n) \quad F(x,r) = \mathcal{M}_r[f(x)]$

$F(x,r) =$ (average of $f(x)$ over sphere of radius r centered at x)

$$F(x,r) = \frac{1}{\omega_n r^{n-1}} \int_{|\xi - x| = r} f(\xi)\, dS(\xi)$$

fcn of $(n+1)$-variables $x_1, \dots, x_n, r$

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$ surface area of unit $(n-1)$-sphere in n-space

$dS(\xi)$ = (cyber) surface area element.

or

$$F(x,r) = \frac{1}{\omega_n} \int_{|\xi| = 1} f(x + r\xi)\, dS(\xi)$$

$\xi' = x + r\xi$

$dS(\xi') = r^{n-1} dS(\xi)$

From second formula for F:

$F(x,r)$ is even : $F(x,r) = F(x,-r)$.

$$\begin{cases} F(x,0) = f(x) & \text{by } \int_{|\xi|=1} dS(\xi) = \omega_n \\ \frac{\partial F}{\partial r}(x,0) = 0 & \text{by } F \text{ even.} \end{cases}$$

f in C^2 $\to$ F in C^2 in $(n+1)$-vars.

Darboux's equation.

$$\frac{\partial F}{\partial r}(x,r) = \frac{1}{\omega_n} \frac{\partial}{\partial r} \int_{|\xi|=1} f(x + r\xi)\, dS(\xi)$$

$$= \frac{1}{\omega_n} \int_{|\xi|=1} \frac{\partial}{\partial r} f(x + r\xi)\, dS(\xi) = \frac{1}{\omega_n} \int_{|\xi|=1} \nabla f(x + r\xi) \cdot \xi\, dS(\xi)$$

(chain rule; ξ: unit normal)

$$= \frac{1}{\omega_n} \int_{|\xi|<1} \nabla_\xi \cdot (\nabla f(x + r\xi))\, dV(\xi)$$ by divergence theorem

($dV(\xi) = d\xi_1 \cdots d\xi_n$)

$$= \frac{1}{\omega_n} \int_{|\xi|<1} (\Delta f)(x + r\xi)\, r\, dV(\xi) = \frac{r}{\omega_n} \int_{|\xi|<1} (\Delta f)(x + r\xi)\, dV(\xi)$$

$$= \frac{r}{\omega_n} \Delta_x \int_{|\xi|<1} f(x+r\xi)\, dV(\xi) .$$

$$\int_{|\xi|<1} f(x+r\xi)\, dV(\xi) = \int_{|\xi-x|<r} f(\xi)\, r^{-n}\, dV(\xi) \quad \Big| \quad \begin{array}{l} \xi' = x + r\xi \\ dV(\xi') = r^n dV(\xi) \end{array}$$

$$= \frac{1}{r^n} \int_{|\xi-x|<r} f(\xi)\, dV(\xi) = \frac{1}{r^n} \int_0^r \left(\int_{|\xi-x|=\rho} f(\xi)\, dS(\xi) \right) d\rho$$

$$dV(\xi) = d\rho\, dS(\xi)$$

$$= \frac{1}{r^n} \int_0^r \omega_n \rho^{n-1} F(x,\rho)\, d\rho \quad \text{by } F(x,\rho) = \frac{1}{\omega_n \rho^{n-1}} \int \cdots .$$

$$= \frac{\omega_n}{r^n} \int_0^r \rho^{n-1} F(x,\rho)\, d\rho .$$

$$\frac{\partial F}{\partial r}(x,r) = \frac{r}{\omega_n} \Delta_x \left(\frac{\omega_n}{r^n} \int_0^r \rho^{n-1} F(x,\rho)\, d\rho \right)$$

$$= \frac{1}{r^{n-1}} \Delta_x \int_0^r \rho^{n-1} F(x,\rho)\, d\rho .$$

$$\frac{\partial}{\partial r} \left(r^{n-1} \frac{\partial F}{\partial r}(x,r) \right) = \Delta_x\, r^{n-1} F(x,r)$$

$$\frac{\partial^2 F}{\partial r^2}(x,r) + \frac{n-1}{r} \frac{\partial F}{\partial r}(x,r) = (\Delta F)(x,r) \qquad \text{Darboux equation.}$$

Thus: $f \in C^2 \to F$ satisfies Darboux and $F(x,0) = f(x)$, $F_r(x,0) = 0$.

Euler - Poisson - Darboux equation

$u = \varphi(x,t)$ solves the same equation:

$$\begin{cases} \Delta u - \frac{1}{c^2} \frac{\partial^2 u}{\partial x^2} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Set:

$$\Phi(x,r,t) = M_r[\varphi(x,t)] = \frac{1}{\omega_n}\int_{|\beta|=1} \varphi(x+r\beta,t)\,dS(\beta)$$

Vrein x as parameter.

and: $F(x,r) = M_r[f(x)]$

$G(x,r) = M_r[g(x)]$.

$$\Delta_x \Phi(x,r,t) = \frac{1}{\omega_n}\Delta_x \int_{|\beta|=1} \varphi(x+r\beta,t)\,dS(\beta)$$

$$= \int_{|\beta|=1} (\Delta\varphi)(x+r\beta,t)\,dS(\beta) = \frac{1}{\omega_n}\int_{|\beta|=1} \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}(x+r\beta,t)\,dS(\beta)$$

(wave eqn)

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{1}{\omega_n}\int_{|\beta|=1}\varphi(x+r\beta)\,dS(\beta) = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}(x,r,t).$$

$$\Phi(x,r,0) = M_r[\varphi(x,0)] = M_r[f(x)] = F(x,r)$$

$$\Phi_t(x,r,0) = M_r[\varphi_t(x,0)] = M_r[g(x)] = G(x,r)$$

Thus Darboux gives:

$u = \Phi(x,r,t)$ is solution of: (Vrein x as a parameter.)

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{n-1}{r}\dfrac{\partial u}{\partial r} - \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = 0 \\ u(x,r,0) = F(x,r) \\ u_t(x,r,0) = G(x,r) \end{cases}$$

EPD eqn

PDE with only r & t as indep. variables.

Wave equation in space

$u = \varphi(x,t)$ solves wave eqn for $u = 3$.

$\Phi(x,r,t) = M_r[\varphi(x,t)]$

F, G etc as above.

$$\frac{\partial^2}{\partial r^2}(r\Phi) = r\Phi_{rr} + 2\Phi_r = r\left(\Phi_{rr} + \frac{2}{r}\Phi_r\right)$$

$$\underset{\text{EPD}}{=} r\,\frac{1}{c^2}\Phi_{tt} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(r\Phi)$$

$u = r\Phi(x,r,t)$ is solution of:

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} - \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = 0 \\ u(x,r,0) = rF(x,r) \\ u_t(x,r,0) = rG(x,r) \end{cases}$$

d'Alembert →

$$r\Phi(x,r,t) = \frac{1}{2}\left((r-ct)F(x,r-ct) + (r+ct)F(x,r+ct)\right) + \frac{1}{2c}\int_{r-ct}^{r+ct} \rho\, G(x,\rho)\, d\rho$$

F & G even in r → $F(x,-r) = F(x,r)$, $rG(x,r)$ odd

$$\Phi(x,r,t) = \frac{(ct+r)F(x,ct+r) - (ct-r)F(x,ct-r)}{2r} + \frac{1}{2cr}\int_{ct-r}^{ct+r} \rho\, G(x,\rho)\, d\rho$$

← by $\int_{ct-r}^{r-ct}(\text{odd}) = 0$

Let $r \to 0$. $\dfrac{f(x+h)-f(x-h)}{2h} \to f'(x)$ as $h \to 0$.

$$\Phi(x,0,t) = \frac{\partial}{\partial r}\left(rF(x,r)\right)_{r=ct} + \frac{1}{2c}\left(ct\, G(x,ct)\right)$$

$$\boxed{\varphi(x,t) = \frac{\partial}{\partial t}\left(t\,F(x,ct)\right) + t\,G(x,ct)}$$

Kirchoff's formula.

Any solution of wave eqn must be given by this formula. → Solution unique.

Check: this above formula gives a solution. → Solution exists.

$$\Delta_x\left(t\,G(x,ct)\right) = t\left(G_{rr} + \frac{2}{r}G_r\right)_{r=ct}$$

Darboux

$$= t\,G_{rr}(x,ct) + \frac{2}{c}G_r(x,ct)$$

$$= t\,\frac{1}{c^2}\frac{\partial^2}{\partial t^2}G(x,ct) + \frac{2}{c}\,\frac{1}{c}\frac{\partial}{\partial t}G(x,ct)$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(t\,G(x,ct)\right).$$

$$\Delta_x \frac{\partial}{\partial t}(t F(x,ct)) = \frac{\partial}{\partial t}\Delta_x(t F(x,ct))$$

$$= \text{as for } G \quad \frac{\partial}{\partial t}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(t F(x,ct)) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial t}(t F(x,ct)).$$

$t=0$:

$$\varphi(x,t) = \frac{\partial}{\partial t}(t F(x,ct)) + t G(x,ct)$$

$$= F(x,ct) + t\frac{\partial}{\partial t}F(x,ct) + t G(x,ct)$$

$$\varphi(x,0) = F(x,0) + 0 = f(x).$$

$$\varphi_t(x,t) = \underbrace{\frac{\partial^2}{\partial t^2}(t F(x,ct))}_{\text{und}} + \frac{\partial}{\partial t}(t G(x,ct))$$

$$\varphi_t(x,0) = 0 + \underbrace{G(x,0)}_{\text{ist bekannt}} = g(x).$$

<u>Kirchoff's formula</u>

$$F(x,r) = \frac{1}{4\pi r^2}\int_{|\xi - x| = r} f(\xi)\, dS(\xi) = \frac{1}{4\pi}\int_{|\xi|=1} f(x + r\xi)\, dS(\xi)$$

($\omega_3 = 4\pi$)

$$t F(x,ct) = \frac{1}{4\pi}\int_{|\xi|=1} t\, f(x + ct\xi)\, dS(\xi)$$

$$\frac{\partial}{\partial t}(t F(x,ct)) = \frac{1}{4\pi}\int_{|\xi|=1}\Big(f(x+ct\xi) + t(\nabla f)(x+ct\xi)\cdot c\xi\Big)\, dS(\xi)$$

Somit φ ist:

$$\varphi(x,t) = \frac{1}{4\pi}\int_{|\xi|=1}\Big(f(x+ct\xi) + t(\nabla f)(x+ct\xi)\cdot c\xi + t\, g(x+ct\xi)\Big)\, dS(\xi)$$

$\left.\begin{aligned} \xi' &= x + ct\xi \\ dS(\xi') &= (ct)^2\, dS(\xi) \end{aligned}\right\} \rightarrow$

$$\varphi(x,t) = \frac{1}{4\pi c^2 t^2} \int_{|\xi - x| = ct} \Big(f(\xi) + (\nabla f)(\xi)\cdot(\xi - x) + t g(\xi) \Big)\, dS(\xi)$$

$f \in C^3$, $g \in C^2$ $\Big\}$ → solution $\varphi \in C^2$.

An integral

$$\int_{x^2+y^2+z^2=a^2} f(x,y,z)\, dS(x,y,z)$$

sphere in space $z = \sqrt{a^2 - x^2 - y^2}$

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{a^2-x^2-y^2}}$$

$$dS(x,y,z) = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\,dy = \frac{a\, dx\, dy}{\sqrt{a^2-x^2-y^2}}$$

$$\int_{x^2+y^2+z^2=a^2} f(x,y,z)\, dS(x,y,z) = \iint_{x^2+y^2\le a} \Big(f\big(x,y,\sqrt{a^2-x^2-y^2}\big) + f\big(x,y,-\sqrt{a^2-x^2-y^2}\big) \Big) \frac{a\, dx\, dy}{\sqrt{a^2-x^2-y^2}}$$

sphere surface with

If f does not depend on z !

$$\int_{\text{sphere}} f(x,y)\, dS(x,y,z) = 2a \iint_{\text{disk}} f(x,y)\, \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}.$$

Wave equation in plane.

$u(x,y,t) = \varphi(x,y,t)$ solution of :

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} - \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = 0 \\ u(x,y,0) = f(x,y) \\ u_t(x,y,0) = g(x,y) \end{cases}$$

Then : $u(x,y,z,t) = \varphi(x,y,t)$ is solution of wave eqn in space and formula apply.

Hadamard Descent.

$$K(x,y,\tau,\xi,\eta) = \frac{1}{c^2\tau^2}\Big(f(\xi,\eta) + \frac{\partial f}{\partial x}(\xi,\eta)(\xi-x) + \frac{\partial f}{\partial y}(\xi,\eta)(\eta-y) + \tau g(\xi,\eta)\Big).$$

Kirchoff & integral formulas $\rightarrow$:

$$\varphi(x,y,\tau) = \frac{1}{4\pi}\int_{(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2 = c^2\tau^2} K(x,y,\tau,\xi,\eta)\, dS(\xi,\eta,\zeta)$$

$$= \frac{1}{4\pi}\cdot 2c\tau \iint_{(\xi-x)^2+(\eta-y)^2 < c^2\tau^2} K(x,y,\tau,\xi,\eta)\, \frac{d\xi\, d\eta}{\sqrt{c^2\tau^2-(\xi-x)^2-(\eta-y)^2}}$$

$$\boxed{\varphi(x,y,\tau) = \frac{1}{2\pi}\iint_{(\xi-x)^2+(\eta-y)^2 < c^2\tau^2} c\tau K(x,y,\tau,\xi,\eta)\, \frac{d\xi\, d\eta}{\sqrt{c^2\tau^2-(\xi-x)^2-(\eta-y)^2}}}$$

Additional formulas

$n=3$

$$F(x,r) = \frac{1}{4\pi r^2}\int_{|\xi-x|=r} f(\xi)\, dS(\xi)$$

$$\tau F(x,c\tau) = \frac{1}{4\pi c^2\tau}\int_{|\xi-x|=c\tau} f(\xi)\, dS(\xi)$$

Kirchoff gives:

$$\boxed{\varphi(x,\tau) = \frac{\partial}{\partial\tau}\left(\frac{1}{4\pi c^2\tau}\int_{|\xi-x|=c\tau} f(\xi)\, dS(\xi)\right) + \frac{1}{4\pi c^2\tau}\int_{|\xi-x|=c\tau} g(\xi)\, dS(\xi)}$$

$x = (x_1,x_2,x_3)$:

$n=2$. Descend from last formula above:

$$\int_{(3-x)^2+(4-y)^2+(z-2)^2=c^2t^2} f(\xi,\eta)\, d\sigma(\xi,\eta,z) = 2\pi c t \int_{(\xi-x)^2+(\eta-y)^2<c^2t^2} f(\xi,\eta)\, \frac{d\xi\, d\eta}{\sqrt{c^2t^2-(\xi-x)^2-(\eta-y)^2}}$$

$$\varphi(x,y,t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi c}\int_{(\xi-x)^2+(\eta-y)^2<c^2t^2} f(\xi,\eta)\, \frac{d\xi\, d\eta}{\sqrt{c^2t^2-(\xi-x)^2-(\eta-y)^2}}\right)$$
$$+ \frac{1}{2\pi c}\int_{(\xi-x)^2+(\eta-y)^2<c^2t^2} g(\xi,\eta)\, \frac{d\xi\, d\eta}{\sqrt{c^2t^2-(\xi-x)^2-(\eta-y)^2}}$$

Poisson's formula.