The wave equation

In one dimension

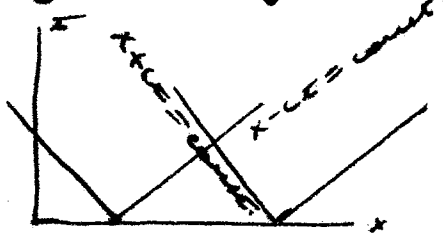
PDE $\boxed{\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0}$

IC
$u(x,0) = f(x)$
$u_t(x,0) = g(x)$

Γ is x-axis
$\frac{\partial u}{\partial n}$ is $u_t(x,0)$ (outward normal).

Characteristics

Solve ODE
$A\,dy^2 - 2B\,dx\,dy + C\,dx^2 = 0.$



$A = 1,\ B = 0,\ C = -\frac{1}{c^2}.$

$dt^2 - \frac{1}{c^2}dx^2 = 0.$

$(dx - c\,dt)(dx + c\,dt) = 0$

$x - ct = \text{const}$
$x + ct = \text{const}$

(OVER)

D'Alembert's solution

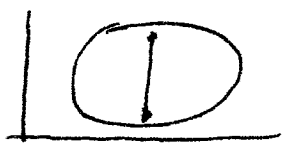
Use characteristic coordinate curves.

$$\begin{cases} \xi = x - ct \\ \eta = x + ct. \end{cases}$$

$$\left.\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta \\ u_{xx} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\ u_t &= u_\xi \xi_t + u_\eta \eta_t = -cu_\xi + cu_\eta \\ u_{tt} &= c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) \end{aligned}\right\} \rightarrow u_{xx} - \frac{1}{c^2}u_{tt} = 4u_{\xi\eta}$$

PDE: $u_{\xi\eta} = 0.$

Assume domain of u is convex.
($\xi\eta$ or xt coords).



$(u_\xi)_\eta = 0 \rightarrow u_\xi(\xi,\eta) = a(\xi)$ independent of η

$u(\xi,\eta) = \int a(\xi)\,d\xi + \left(\begin{matrix}\text{constant} \\ \text{dep. on } \eta\end{matrix}\right)$

$u = A(\xi) + B(\eta).$

For characteristic condition:

$x = x_0 \pm ct$ characteristic

$\psi(x,t) = (x-x_0)^2 - c^2t^2$

$\psi_x^2 - \frac{1}{c^2}\psi_t^2 = 4\left((x-x_0)^2 - c^2t^2\right) = 0$ on $\psi = 0$

$\psi(x,t) = (x-x_0) \mp ct$

$\psi_x^2 - \frac{1}{c^2}\psi_t^2 = 0$ always.



$u = A(x-ct) + B(x+ct)$

$u(x,0) = f(x) \longrightarrow A(x) + B(x) = f(x)$

$u_t(x,0) = g(x) \longrightarrow -cA'(x) + cB'(x) = g(x)$

$$\begin{cases} A'(x) + B'(x) = f'(x) \\ -A'(x) + B'(x) = \frac{g(x)}{c} \end{cases} \longrightarrow \begin{cases} A' = \frac{1}{2}\left(f' - \frac{g}{c}\right) \\ B' = \frac{1}{2}\left(f' + \frac{g}{c}\right) \end{cases}$$

$$\begin{cases} A(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(\xi)\,d\xi + a \\ B(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\xi)\,d\xi + b \end{cases}$$

a and b constants.

$A(x) + B(x) = f(x) \longrightarrow a + b = 0$

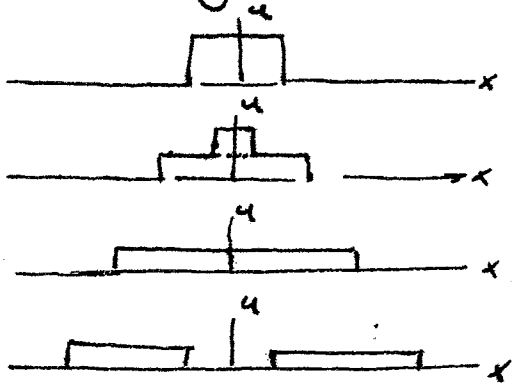
$$\boxed{u = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi)\,d\xi}$$

d'Alembert formula.

Note: For f in C^2 and g in C^1 there is a unique u in C^2.

Note. $u = A(x-ct) + B(x+t)$
is: $u =$ (wave to right) + (wave to left).

Ex For $g \equiv 0$.



Increasing time t.

Boundary conditions. Example. Smirnov Vol. 2, 15 ff.

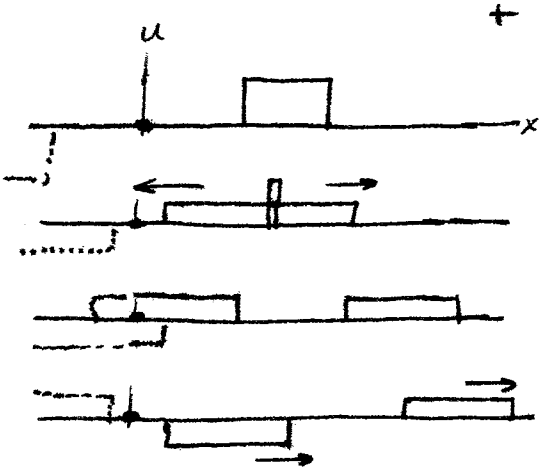
I.C.
$$u(x,0) = \begin{cases} f(x) & x \geq 0 \\ 0 & x < 0 \end{cases}$$
$$u_t(x,0) = \begin{cases} g(x) & x \geq 0 \\ 0 & x < 0 \end{cases}$$

B.C.
$$u(0,t) = 0 \quad t \geq 0$$
$$u_x(0,t) = 0 \quad t \geq 0$$

Reflection f and g to be odd functions! $\begin{cases} f(-x) = -f(x) \\ g(-x) = -g(x). \end{cases}$

d'Alembert formula gives solution. For:

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\left(-cf'(x-ct) + cf'(x+ct)\right) + \frac{1}{2}\left(g(x+ct) + g(x-ct)\right)$$

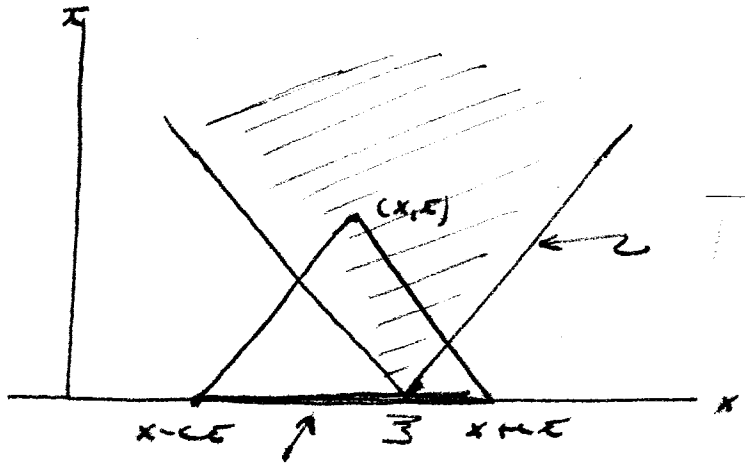


Similarly: The BCs mean

$$\begin{cases} u(0,t) = 0 \\ u_x(0,t) = 0 \\ u(L,t) = 0 \\ u_x(L,t) = 0 \end{cases} \quad t \geq 0$$

by extending f & g to be periodic with period $2L$.

Domain of dependence, ~~and influence~~.



Domain (interval) of dependence of solution at point (x,t).

Range of influence of point 3 on solution.

i.e. Disturbance or signal propagate with speed c.

Uniqueness of solutions

$$\begin{cases} u_{xx} - \frac{1}{c^2} u_{tt} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

$$\begin{cases} u(A,t) = a(t) \\ u(B,t) = b(t) \end{cases} \quad t \geq 0$$

$\varphi(x,t)$ = difference between two c^2 sol'ns.

$$\begin{cases} \varphi_{xx} - \frac{1}{c^2} \varphi_{tt} = 0 \\ \varphi(x,0) = 0 \\ \varphi_t(x,0) = 0 \end{cases}$$

$$\begin{cases} \varphi(A,t) = 0 \\ \varphi(B,t) = 0 \end{cases} \quad t \geq 0$$

Energy:

$$E(t) = \frac{1}{2} \int_A^B \left(\varphi_x(x,t)^2 + \frac{1}{c^2} \varphi_t(x,t)^2 \right) dx$$

$$\frac{dE}{dt} = \int_A^B \left(\varphi_x \varphi_{xt} + \frac{1}{c^2} \varphi_t \varphi_{tt} \right) dx$$

$$(\varphi_x \varphi_t)_x = \varphi_{xx} \varphi_t + \varphi_x \varphi_{xt}$$

$$= \int_A^B \left((\varphi_x \varphi_t)_x - \varphi_{xx} \varphi_t + \frac{1}{c^2} \varphi_t \varphi_{tt} \right) dx$$

$$= \underbrace{\left[\varphi_x \varphi_t \right]_A^B}_{0} - \int_A^B \underbrace{\left(\varphi_{xx} - \frac{1}{c^2} \varphi_{tt} \right)}_{0} \varphi_t \, dx = 0.$$

So $E(t)$ = constant.

$$\left.\begin{aligned} \varphi_t(A,0) &= 0 \\ \varphi_t(B,0) &= 0 \end{aligned}\right\} \longrightarrow E(t) = 0 \quad \text{all } t.$$

$$E(t) \equiv 0 \rightarrow \begin{cases} \varphi_x \equiv 0 \\ \varphi_t \equiv 0 \end{cases} \rightarrow \varphi = \text{const} \rightarrow \varphi \equiv 0.$$

i.e. $\varphi(x,t) = 0 \quad A \leq x \leq B \quad$ & $\quad t \geq 0.$

A & B arbitrary $\rightarrow$ $\varphi(x,t) = 0$ all x & all $t \geq 0$.