Wave fronts and rays

PDE $\sum_{i,j=1}^{4} A^{ij}(x) \frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial t^2} = 0$, $x = (x^1, \dots, x^4)$

Charac. condtn $\sum_{i,j=1}^{4} A^{ij}(x) \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j} - \left(\frac{\partial \psi}{\partial t}\right)^2 = 0$ on $\psi(x,t) = 0$.

Charac. condition crucial! Charac. hypersurface of form $t = \varphi(x^1, \dots, x^4)$.

$\psi(x,t) = \varphi(x) - t \longrightarrow \sum_{i,j=1}^{4} A^{ij}(x) \frac{\partial \varphi}{\partial x^i} \frac{\partial \varphi}{\partial x^j} = 1$.

PDE for charac. hypersurfaces: $\sum_{i,j=1}^{4} A^{ij}(x) \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} = 1$.

$u = \varphi(x)$ solns $\longrightarrow$ $\begin{cases} \varphi(x) - t = 0 \text{ is} \\ \text{charac. hypersurface.} \end{cases}$

Interpretation. Surface of discontinuity of soln of PDE moves through x space with t. Cf Hamilton equation & wave fronts.

Bicharacteristic CEs for PDE! $\sum_{i,j=1}^{4} A^{ij}(x) \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} - \left(\frac{\partial u}{\partial t}\right)^2 = 0$.

$p_i \leftrightarrow \frac{\partial u}{\partial x^i}$

$q \leftrightarrow \frac{\partial u}{\partial t}$

$F = \frac{1}{2}\left(\sum A^{ij} p_i p_j - q^2\right)$

$$\begin{cases} \frac{dx^\alpha}{d\tau} = \sum A^{\alpha\beta}(x) p_\beta \\ \frac{dt}{d\tau} = -q \\ \frac{du}{d\tau} = \sum A^{\alpha\beta} p_\alpha p_\beta - q^2 = 2F = 0 \text{ on } F = 0 \\ \frac{dp_\alpha}{d\tau} = -\frac{1}{2} \sum \frac{\partial A^{\alpha\beta}}{\partial x^\alpha}(x) p_\alpha p_\beta \\ \frac{dq}{d\tau} = 0 \end{cases}$$

$5^{th} \rightarrow q = \text{const.}$
$2^{nd} \rightarrow \tau = -q\tau + \text{const}$ } → Take $\tau = \tau$ as parameter along any char. c.

Remaining equations:

$$\left.\begin{aligned} \frac{dx^\alpha}{d\tau} &= \sum A^{i\beta} p_\beta \\ \frac{dp_i}{d\tau} &= -\frac{1}{2}\sum \frac{\partial A^{\alpha\beta}}{\partial x^i} p_\alpha p_\beta \end{aligned}\right\} \quad \begin{aligned} x^i &= X^i(\tau) \\ p_i &= P_i(\tau) \end{aligned} \quad \text{solve in } xp\text{-space.}$$

$$\frac{dq}{d\tau} = \sum A^{i\beta} p_i p_\beta - 1 = 0$$

CEs for the PDE: $\sum_{i,j=1}^{n} A^{ij}(x) \frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j} = 1$, $F = \frac{1}{2}\left(\sum A^{ij} p_i p_j - 1\right)$.

$$\begin{cases} \frac{dx^i}{d\tau} = \sum A^{i\beta} p_\beta \\ \frac{du}{d\tau} = \sum A^{i\beta} p_i p_\beta = 1 \\ \frac{dp_i}{d\tau} = -\frac{1}{2}\sum \frac{\partial A^{\alpha\beta}}{\partial x^i} p_\alpha p_\beta \end{cases} \qquad \begin{cases} x^i = X^i(\tau) \\ u = \tau + c \\ p_i = P_i(\tau) \end{cases}$$

Soln. in xp & xu above. $c = \text{const.}$

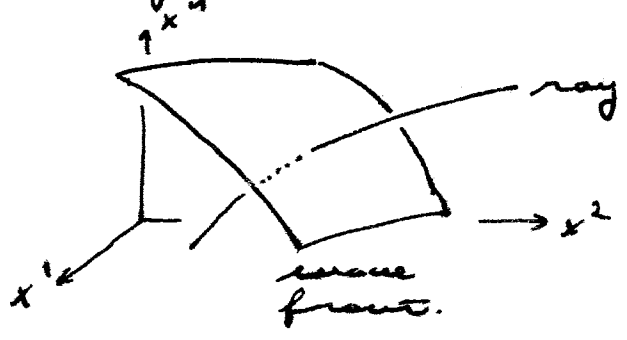
Thus: In $x\tau$ space:

$$\left(\begin{array}{c}\text{bicharacteristics of} \\ \sum A^{ij}\frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial \tau^2} = 0\end{array}\right) = \left(\begin{array}{c}\text{characteristics of} \\ \sum A^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j} = 1\end{array}\right).$$

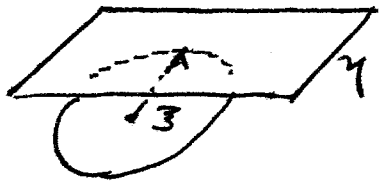
Terminology: In x-space:

(characteristic hypersurface) ↔ (wave front (depends on τ))

(bicharacteristic curve) ↔ ray



Conjugate directions



Quadric: $\sum_{i,j=1}^{n} A_{ij} x^i x^j = 1$

Tangent hyperplane to quadric at Σ: $\sum A^{\Sigma}_{ij} \Sigma^i x^j = 1$. $\leftrightarrow$ Point η (of euclidean space): $\sum \eta_j x^j = 1$.

$\rightarrow \eta_j = \sum A_{ij} \Sigma^i$ or $\Sigma^i = \sum A^{ij} \eta_j$ $\quad | \quad (A^{ij}) = (A_{ij})^{-1}$.

(OVER)

Point Σ & plane $\sum \eta_j x^j = 1$ are conjugate with respect to the quadric.

Wave and ray directions

$u = \varphi(x)$ solution of $\sum A^{ij}(x) \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} = 1$

$p_i = \frac{\partial \varphi}{\partial x}(x)$ slip.

$\varphi(x) = t + c$ wave front at time t.

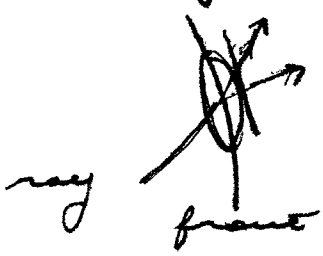
$$\begin{cases} x^i = X^i(\tau) & \text{Characteristic with} \\ u = \tau + c & \text{same parameter } \tau \\ p_i = P_i(\tau) & \text{- as above.} \end{cases}$$

First & second eqs above:

$$\begin{cases} \frac{dX^i}{d\tau}(\tau) = \sum A^{i\beta}(X(\tau)) \frac{\partial \varphi}{\partial x^\beta}(X(\tau)) \\ \sum A^{i\beta}(X(\tau)) \frac{\partial \varphi}{\partial x^i}(X(\tau)) \frac{\partial \varphi}{\partial x^\beta}(X(\tau)) = 1 \end{cases}$$

$$\begin{cases} \frac{dX^i}{d\tau} \leftrightarrow \Sigma^i \\ \frac{\partial \varphi}{\partial x^i} \leftrightarrow \eta_i \end{cases}$$

Ray direction and tangent plane to wave front are conjugate with respect to quadric (dual) $\sum A^{ij}(X(\tau)) \eta_i \eta_j = 1$.



C.H. II. p 560 ff.
Analyt. Mechanics. p. 248 ff.

$$\sum A_{ij} \eta_i \eta_j = 1 \quad \text{by} \quad \sum A^{ij} \eta_i \eta_j = \sum A^{ij} A_{i\alpha} \xi^\alpha A_{j\beta} \xi^\beta$$
$$= \sum \delta^j_\alpha \xi^\alpha A_{j\beta} \xi^\beta = \sum A_{j\beta} \xi^j \xi^\beta$$