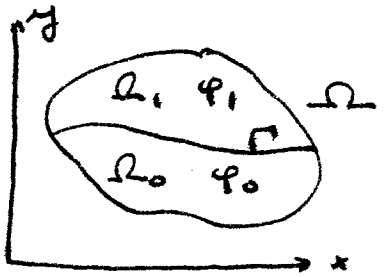
Propagation of singularities

F.J. pp. 33 ff.

Jumps



Ω_0 & Ω_1 open & disjoint

Γ : $\begin{cases} x = \xi(s) \\ y = \eta(s) \end{cases}$

φ_0 C^1 on nbhd of $\Omega_0 + \Gamma$

φ_1 C^1 —— " —— $\Omega_1 + \Gamma$.

$\varphi = \begin{cases} \varphi_0 & \text{on } \Omega_0 \\ \varphi_1 & \text{on } \Omega_1 \end{cases}$ has jump discontinuity along Γ.

$[\varphi] = \varphi_1(\xi(s+1), \eta(s+1)) - \varphi_0(\xi(s), \eta(s))$ jump at point $(\xi(s), \eta(s+1))$ of Γ.

$$\frac{d}{ds}[\varphi] = \varphi_{1x}\xi' + \varphi_{1y}\eta' - \varphi_{0x}\xi' - \varphi_{0y}\eta'$$
$$= [\varphi_x]\xi' + [\varphi_y]\eta'.$$

Theorem: φ continuous on $\Omega \rightarrow$
$[\varphi] = 0$ and $[\varphi_x]\xi' + [\varphi_y]\eta' = 0$.

φ_0 C^2 on nbhd of $\Omega_0 + \Gamma$
φ_1 C^2 —— " —— $\Omega_1 + \Gamma$
$\varphi = \begin{cases} \varphi_0 & \text{on } \Omega_0 \\ \varphi_1 & \text{on } \Omega_1 \end{cases}$ C^1 on Ω

$\rightarrow$ $[\varphi_x] = 0$; $[\varphi_y] = 0$

$[\varphi_{xx}]\xi' + [\varphi_{xy}]\eta' = 0$

$[\varphi_{xy}]\xi' + [\varphi_{yy}]\eta' = 0$.

Quasi-linear PDE

$A u_{xx} + 2B u_{xy} + C u_{yy} = D$, $A = A(x, y, u, u_x, u_y)$, etc.

φ_0 C^2 soln of PDE on nbhd of $\Omega_0 + \Gamma$
φ_1 C^2 —— " —— " —— $\Omega_1 + \Gamma$

$\varphi = \begin{cases} \varphi_0 & \text{on } \Omega_0 \\ \varphi_1 & \text{on } \Omega_1 \end{cases}$ C^1 on Ω

$A(x, y, \varphi_0, \varphi_{0x}, \varphi_{0y})\varphi_{0xx} + \cdots = D(\cdots)$ $\quad \varphi_{0x} = \varphi_{1x}$

$A(x, y, \varphi_1, \varphi_{1x}, \varphi_{1y})\varphi_{1xx} + \cdots = D(\cdots)$ $\quad \varphi_{0y} = \varphi_{1y}$

$\rightarrow$ $A[\varphi_{xx}] + 2B[\varphi_{xy}] + C[\varphi_{yy}] = 0$.

Assume not all $[\varphi_{xx}]$, $[\varphi_{xy}]$, $[\varphi_{yy}]$ are 0.

$$\begin{cases} A[\varphi_{xx}] + 2B[\varphi_{xy}] + C[\varphi_{yy}] = 0 \\ \xi'[\varphi_{xx}] + \eta'[\varphi_{xy}] = 0 \\ \xi'[\varphi_{xy}] + \eta'[\varphi_{yy}] = 0 \end{cases}$$ has non-trivial solution.

$$\rightarrow \begin{vmatrix} A & 2B & C \\ \xi' & \eta' & 0 \\ 0 & \xi' & \eta' \end{vmatrix} = A\eta'^2 - 2B\xi'\eta' + C\xi'^2 = 0.$$

Γ is a charac. curve.

Thus: Singularities in the second derivatives of solutions of the PDE lie on characteristics.

Second simplement canonical form

Change coords so $\Gamma^{(n-1)}$ is $y^n = 0$ as before.

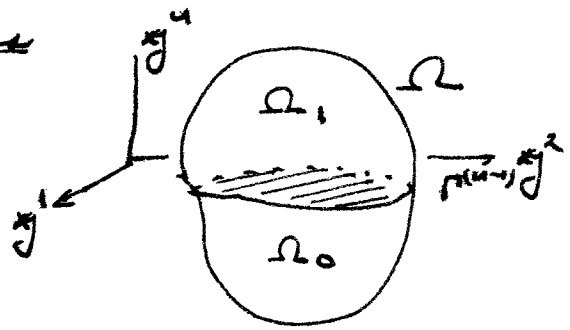
PDE: $\sum B^{\alpha\beta} \frac{\partial^2 u}{\partial y^\alpha \partial y^\beta} = E$.

$B^{\alpha\beta} = B^{\alpha\beta}(y, u, u_y)$, etc.

φ_0 C^2 on nbhd of $\Omega_0 + \Gamma$

φ_1 C^2 —— " —— $\Omega_1 + \Gamma$

φ = $\begin{cases} \varphi_0 \text{ on } \Omega_0 \\ \varphi_1 \text{ on } \Omega_1 \end{cases}$ C^1 on Ω



For any f on Γ: $[f] = f_1(y^1, \cdots, y^{n-1}, 0) - f_0(y^1, \cdots y^{n-1}, 0)$.

so $\frac{\partial}{\partial y^\alpha}[f] = \left[\frac{\partial f}{\partial y^\alpha}\right]$ $\alpha = 1, 2, \cdots, n-1$.

$\left[\frac{\partial \varphi}{\partial y^\beta}\right] = 0$ $\beta = 1, 2, \cdots, n$ $\rightarrow$ $\left[\frac{\partial^2 \varphi}{\partial y^\alpha \partial y^\beta}\right] = 0$ $\alpha = 1, 2, \cdots n-1$, $\beta = 1, 2, \cdots n$.

We only $\left[\frac{\partial^2 \varphi}{\partial y^{n2}}\right]$ can be $\neq 0$.

$\sum B^{\alpha\beta} \frac{\partial^2 \varphi_1}{\partial y^\alpha \partial y^\beta} = E$ fits for φ_0 $\Big\} \rightarrow$ subtract $B^{nn}(y, \varphi, \varphi_y)\left[\frac{\partial^2 \varphi}{\partial y^{n2}}\right] = 0$.

If a jump occurs, $\left[\frac{\partial^2 \varphi}{\partial y^{n2}}\right] \neq 0$ and so $B^{nn} = 0$.

Γ is necessarily a characteristic hypersurface.