Characteristics

F.J. 6631 ff.

Quasi-linear equations

PDE $\begin{cases} A\,u_{xx} + 2B\,u_{xy} + C\,u_{yy} = D \\ A, B, C, D \text{ functions of } x, y, u, u_x, u_y \text{ only.} \end{cases}$ $\Rightarrow A = A(x, y, u, u_x, u_y)$ etc.

Cauchy data: $\Gamma \begin{cases} x = x_0(t) \\ y = y_0(t) \end{cases}$

$\begin{cases} u = u_0(t) \\ u_x = a_0(t) \\ u_y = b_0(t) \end{cases}$ satisfying $u_0' = a_0 x_0' + b_0 y_0'$.

Motivation Use PDE to determine higher derivatives of solution on Γ in order to obtain (formal) power series solution.

Restrictions on higher derivatives of solutions.

$u = \varphi(x, y)$ solves PDE & Cauchy data.

$\varphi_x(x_0(t), y_0(t)) = a_0(t)$ and derivative $\longrightarrow \begin{cases} \varphi_{xx} x_0' + \varphi_{xy} y_0' = a_0' \\ \varphi_{xy} x_0' + \varphi_{yy} y_0' = b_0' \end{cases}$ ← at $(x_0(t), y_0(t))$.

Similarly: φ_{xx} known on Γ:

$\varphi_{xx}(x_0(t), y_0(t)) =$ known func of t

$\rightarrow \varphi_{xxx} x_0' + \varphi_{xxy} y_0' = \frac{d}{dt}$ (—— " ——)

a restriction on φ_{xxx} and φ_{xxy} on Γ.

And so on.

Determination: Of second derivatives on Γ.

$A(x_0(t), y_0(t), u_0(t), a_0(t), b_0(t))$, etc.

$$\begin{cases} A\,u_{xx} + 2B\,u_{xy} + C\,u_{yy} = D \\ x_0'\,u_{xx} + y_0'\,u_{xy} = a_0' \\ x_0'\,u_{xy} + y_0'\,u_{yy} = b_0' \end{cases}$$

Solve linear eqns for u_{xx}, u_{xy}, u_{yy}.

Need $\Delta(t) \neq 0$ to have nonsingular solutions.

$$\Delta(t) = \begin{vmatrix} A & 2B & C \\ x_0' & y_0' & 0 \\ 0 & x_0' & y_0' \end{vmatrix} = A y_0'^2 - 2B x_0' y_0' + C x_0'^2 .$$

Of higher derivatives on Γ. Eg Third derivs.

$$A u_{xxx} + 2B u_{xxy} + C u_{yyy} = D$$

$\frac{\partial}{\partial x}$: $A u_{xxxx} + (A_x + A_u u_x + A_{u_x} u_{xx} + A_{u_y} u_{xy}) u_{xx}$
$+ \text{etc.} = \text{etc.}$

$$\rightarrow \quad A u_{xxxx} + 2B u_{xxxy} + C u_{xyyy} = G$$

for of $x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}$
evaluated on Γ

$$\begin{cases} A u_{xxxx} + 2B u_{xxxy} + C u_{xyyy} = G \\ x_0' u_{xxxx} + y_0' u_{xxxy} = \text{known on } \Gamma \\ x_0' u_{xxxy} + y_0' u_{xyyy} = \text{known on } \Gamma \end{cases}$$

Solve for u_{xxxx}, u_{xxxy}, u_{xyyy}.
Need $\Delta(t) \neq 0$. Same $\Delta(t)$ as above.

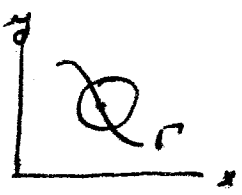
Formal series solution

Γ non-characteristic : $\Delta(t) \neq 0$ on Γ.

$a_{m,n}(t)$ $\quad \frac{\partial^{m+n} u}{\partial x^m \partial y^n}(x_0(t), y_0(t))$ known

$$u = \sum_{m,n=0}^{\infty} \frac{a_{m,n}(t)}{m!\,n!} (x - x_0(t))^m (y - y_0(t))^n$$

- series in neighborhood of a point $(x_0(t), y_0(t))$ of Γ.



Cauchy - Kowalewski Theorem: If PDE & Cauchy data is analytic, so is solution.
i.e. Above power series provides solution near Γ.

Characteristics: Γ such that $\Delta(\tau) \equiv 0$.
Γ can not be used to give Cauchy data.
Linear eqns for u_{xx} etc. either have no solutions or solutions are not unique.

For Γ given implicitly: $\psi(x,y) = 0$.

$\psi(x_0(\tau), y_0(\tau)) = 0$
$\psi_x x_0' + \psi_y y_0' = 0$.

$y_0' = -\frac{\psi_x}{\psi_y} x_0'$: $A y_0'^2 - 2B x_0' y_0' + C x_0'^2$
$= (A\psi_x^2 + 2B\psi_x\psi_y + C\psi_y^2) \left(\frac{x_0'}{\psi_y}\right)^2$

Thus: Characteristics $\psi(x,y) = 0$ satisfy:

Characteristic condition: $A\psi_x^2 + 2B\psi_x\psi_y + C\psi_y^2 = 0$ on $\psi = 0$.

Note! $A = A(x, y, u, u_x, u_y)$, etc., so characs. depend on Cauchy data or solution of the PDE.

Linear equations

PDE $\begin{cases} A u_{xx} + 2B u_{xy} + C u_{yy} + 2D u_x + 2E u_y + F u = G \\ A, B, \cdots, G \text{ functions of } x \text{ and } y \text{ only.} \\ \text{i.e. } A = A(x,y), \text{ etc.} \end{cases}$

Note! Sometimes $D = D(x, y, u, u_x, u_y)$ is meant $-2Du_x - 2Eu_y - Fu + G$.

Characteristics

Γ: $x = x_0(\tau)$, $y = y_0(\tau)$ $\longrightarrow$ ODE for curve Γ:

$A y_0'^2 - 2B x_0' y_0' + C x_0'^2 = 0$ $\qquad A\,dy^2 - 2B\,dx\,dy + C\,dx^2 = 0$.

$A = A(x,y)$, etc.

Characs. do not depend on solution or data.

Classification

Discriminant	Type	Characs. through a pt.
$AC - B^2 < 0$	Hyperbolic	Two.
$AC - B^2 = 0$	Parabolic	One.
$AC - B^2 > 0$	Elliptic	None.

Note! Depends on point. Type may change with x, y.

Several independent variables

PDE: $\sum_{i,j=1}^{n} A^{ij}(x,u,u_x)\frac{\partial^2 u}{\partial x^i \partial x^j} = D(x,u,u_x)$. $A^{ij}=A^{ji}$

Data: $\begin{cases} u = u_0 \\ \frac{\partial u}{\partial n} = a_0 \end{cases}$ given func on $\Gamma^{(n-1)}$ $\quad\Big|\quad \Gamma : \psi(x^1,\cdots,x^n)=0$

Change coordinates: $y^\alpha = y^\alpha(x^1,\cdots,x^n)$, $\alpha=1,\cdots,n$

to $\Gamma^{(n-1)}$ is $y^n=0$.

$$\frac{\partial u}{\partial x^i} = \sum \frac{\partial u}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^i}$$

$$\frac{\partial^2 u}{\partial x^i \partial x^j} = \sum \frac{\partial^2 u}{\partial y^\alpha \partial y^\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + \sum \frac{\partial u}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j}$$

PDE: $\sum_{\alpha,\beta=1}^{n} B^{\alpha\beta}(y,u,u_y)\frac{\partial^2 u}{\partial y^\alpha \partial y^\beta} = E(y,u,u_y)$

$$B^{\alpha\beta} = \sum A^{ij}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}$$

Data: $\begin{cases} u(y^1,\cdots,y^{n-1},0) = f(y^1,\cdots,y^{n-1}) \\ \frac{\partial u}{\partial y^n}(y^1,\cdots,y^{n-1},0) = g(y^1,\cdots,y^{n-1}) \end{cases}$ Normal derivative.

Known on $\Gamma^{(n-1)}$: $y^n=0$ is:

$$\frac{\partial u}{\partial y^\alpha} = \frac{\partial f}{\partial y^\alpha} \quad \alpha=1,2,\cdots,n-1 \quad \text{and} \quad \frac{\partial u}{\partial y^n} = g.$$

and known is:

$$\begin{cases} \frac{\partial^2 u}{\partial y^\alpha \partial y^\beta} = \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta} & \alpha,\beta = 1,2,\cdots n-1 \\ \frac{\partial^2 u}{\partial y^\alpha \partial y^n} = \frac{\partial g}{\partial y^\alpha} & \alpha = 1,2,\cdots n-1. \end{cases}$$

Only $\frac{\partial^2 u}{\partial y^{n2}}$ is not known.

Use PDE to determine $\frac{\partial^2 u}{\partial y^{n2}}$.

$$\sum_{(\alpha,\beta)\neq(n,n)} B^{\alpha\beta}\frac{\partial^2 y}{\partial y^\alpha \partial y^\beta} + B^{nn}\frac{\partial^2 u}{\partial y^{n2}} = E. \qquad y^n = \psi(x^1,\cdots,x^n)$$

Need $B^{nn}\neq 0$ to solve.

For characteristics: $B^{nn} = \sum A^{ij}\frac{\partial y^n}{\partial x^i}\frac{\partial y^n}{\partial x^j} = 0.$

Charac. condition for hypersurface $\Gamma^{(n-1)}$:

$$\boxed{\sum_{i,j=1}^{n} A^{ij}(x,u,u_x)\frac{\partial \psi}{\partial x^i}\frac{\partial \psi}{\partial x^j} = 0 \quad \text{on} \quad \psi(x^1,\cdots,x^n)=0}$$

Linear equations

$$\sum_{i,j=1}^{n} A^{ij}(x) \frac{\partial^2 u}{\partial x^i \partial x^j} + 2 \sum_{i=1}^{n} D^i(x) \frac{\partial u}{\partial x^i} + F(x) u = G(x)$$

Characteristic condition: $\sum_{i,j=1}^{n} A^{ij}(x) \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^j} = 0$ on $\psi(x) = 0$.

PDE for charac. hypersurface: $\sum_{i,j=1}^{n} A^{ij}(x) \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} = 0$.

$u = \varphi(x)$ soln $\rightarrow$ $\begin{cases} \varphi(x^1, \dots, x^n) = c \\ \text{charac. hypersurface} \\ \text{for every } c \end{cases}$

For: $\varphi - c$ satisfies charac. condition on $\varphi - c = 0$.

Bicharacteristic

Charac. or charac. strips of PDE chains.

$$F = \frac{1}{2} \sum_{i,j=1}^{n} A^{ij}(x) b_i b_j.$$

ODEs for bicharacteristic curves:

$$\begin{cases} \dfrac{dx^i}{dt} = \dfrac{\partial F}{\partial b_i} = \sum A^{i\beta}(x) b_\beta, & i = 1, 2, \dots, n \\ \dfrac{du}{dt} = \sum b_i \dfrac{\partial F}{\partial b_i} = 2F = 0 & \text{on } F = 0 \\ \dfrac{db_\nu}{dt} = -\dfrac{\partial F}{\partial x^i} - b_i \dfrac{\partial F}{\partial u} = -\dfrac{1}{2} \sum \dfrac{\partial A^{\alpha\beta}}{\partial x^i}(x) b_\alpha b_\beta \end{cases}$$