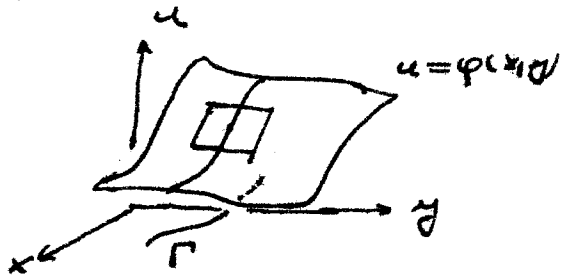
Second order PDEs

PDE $F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0$

Cauchy data

Γ $\begin{cases} x = x_0(t) \\ y = y_0(t) \end{cases}$ Curve in xy-plane.

$\begin{cases} u = u_0(t) \\ u_x = a_0(t) \\ u_y = b_0(t) \end{cases}$ Prescribe u, u_x, u_y on Γ.



Solution $u = \varphi(x, y)$

$$F(x, y, \varphi(x,y), \varphi_x, \varphi_y, \varphi_{xx}, \varphi_{xy}, \varphi_{yy}) = 0$$

and $\begin{cases} \varphi(x_0(t), y_0(t)) = u_0(t) \\ \varphi_x(x_0(t), y_0(t)) = a_0(t) \\ \varphi_y(x_0(t), y_0(t)) = b_0(t). \end{cases}$

Restriction on Cauchy data. $u = \varphi(x,y)$ any solution.

$\varphi(x_0(t), y_0(t)) = u_0(t) \longrightarrow \varphi_x x_0' + \varphi_y y_0' = u_0'$ at $x_0(t), y_0(t)$

$u_0'(t) = a_0(t) x_0'(t) + b_0(t) y_0'(t)$ Cf. Stokes condition.

Compatibility condition.

Suppose we are given Γ and two of u_0, a_0, b_0.

Normal derivative.

Γ $\begin{cases} x = x_0(t) \\ y = y_0(t) \end{cases}$ $f(x,y)$ any function.

$$\begin{pmatrix} \text{tangential} \\ \text{derivative} \\ \text{along } \Gamma \end{pmatrix} = \frac{f_x x_0' + f_y y_0'}{\sqrt{x_0'^2 + y_0'^2}} \qquad f_x = f_x(x_0(t), y_0(t)) \text{ etc.}$$

$$\begin{pmatrix} \text{normal} \\ \text{derivative} \\ \text{along } \Gamma \end{pmatrix} = \frac{-f_x y_0' + f_y x_0'}{\sqrt{\quad\cdots\quad}} = \frac{\partial f}{\partial n}$$

Given Γ and $u_0(t)$: Restriction above

$$\longrightarrow \begin{pmatrix} \text{tangential} \\ \text{derivative} \\ \text{of any} \\ \text{solution} \end{pmatrix} = \frac{a_0 x_0' + b_0 y_0'}{\sqrt{x_0'^2 + y_0'^2}} = \frac{u_0'}{\sqrt{\quad u \quad}}$$

i.e. u_0 prescribes the tangential derivative.

Not determined by u_0 & the normal derivative of solutions along Γ.

Thus: Cauchy data. Solve!

$\Gamma \begin{cases} x = x_0(t) \\ y = y_0(t) \\ u = u_0(t) \\ \frac{\partial u}{\partial n} = v_0(t) \end{cases}$

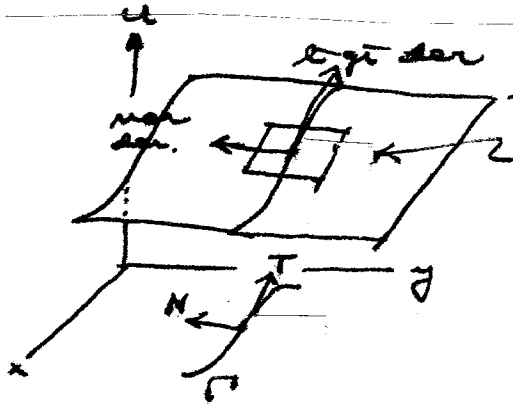
$u_0' = a_0 x_0' + b_0 y_0'$

$v_0 \sqrt{x_0'^2 + y_0'^2} = -a_0 y_0' + b_0 x_0'$

for $a_0(t)$ & $b_0(t)$.

(a_0 & b_0 subject to restriction) $\longleftrightarrow$ v_0

Aside: To give tangential derivatives on Γ is to prescribe $u_0'(t)$ and thus $u_0 = \int u_0' \, dt$.



Tangent plane to solution surface.

PDE: Restrict choices for second & higher derivatives of solutions on Γ of solutions. Eg. Second derivatives must on Γ must satisfy:

$F(x_0, y_0, u_0, a_0, b_0, u_{xx}, u_{xy}, u_{yy}) = 0.$

Several independent variables

PDE: $F\left(x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) = 0$ u scalar.

$\begin{cases} x = (x_1, \cdots, x_n) \\ \frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right) \end{cases}$ and $\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{1 \le i,j \le n}$.

Cauchy data.

$\Gamma^{(n-1)}$: $x = x_0(t)$, $t = (t_1, \cdots, t_{n-1})$, hypersurface

$\begin{cases} u = u_0(t) \\ u_x = a_0(t) \end{cases}$ Prescribe u and u_x on Γ.

Solution: $u = \varphi(x)$.

$F(x, \varphi, \varphi_x, \varphi_{xx}) = 0$ and $\begin{cases} \varphi(x_0(t)) = u_0(t) \\ \varphi_x(x_0(t)) = a_0(t) \end{cases}$

Restriction.

$$\varphi(x_0(t)) = u_0(t) \longrightarrow \sum_{i=1}^{n} \varphi_{x_i}(x_0(t)) \frac{\partial x_{0i}}{\partial t_j}(t) = \frac{\partial u_0}{\partial t_j}(t) \qquad j = 1, 2, \dots, n-1.$$

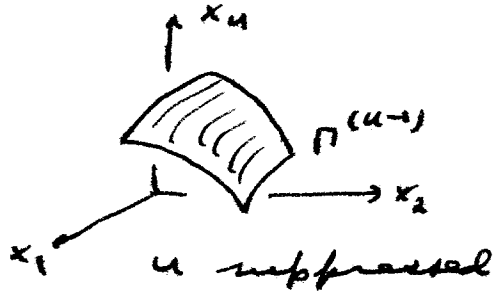
$$\frac{\partial u_0}{\partial t_j}(t) = \sum_{i=1}^{n} a_{0i}(t) \frac{\partial x_{0i}}{\partial t_j}(t) \qquad j = 1, 2, \dots, n-1.$$

Thus we find: $\begin{cases} \Gamma^{(n-1)}: & x = x_0(t) \\ \text{and} & u = u_0(t) \end{cases}$

and i) n functions $a_0(t) = (a_{01}(t), \dots, a_{0n}(t))$ which $n-1$ constraints above

or ii) the normal derivative of u on Γ:

$$\frac{\partial u}{\partial n} = \nabla u \cdot \left(\begin{matrix} \text{normal to} \\ \Gamma \text{ at } x_0(t) \end{matrix} \right).$$



u unknown