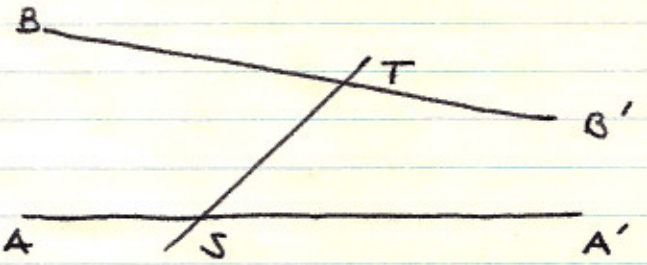
2. The Fifth Postulate

Read: Text. Ch. II.

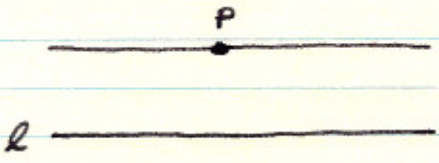
Euclid's and Hilbert's forms of the postulate

Euclid (Fifth Postulate). That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

That is:
If $\angle B'TS + \angle A'ST < 2\rho$, then AA' and BB' meet in the direction of A' and B'.



Hilbert (Playfair's Axiom). Given a line l and a point P not lying on l, there exists, in the plane determined by l and P, one, and only one, line which contains P but not any point of l.



Note: Lines not meeting will be called parallel.

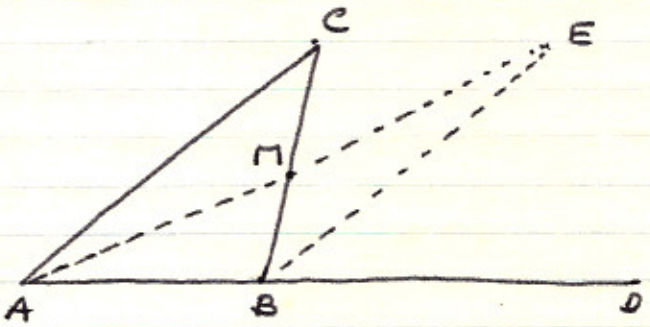
Next step: The equivalence of these two forms.

Note: Assume all of Hilbert's postulates except II, the Parallel Postulate.

Preliminaries to proof of equivalence.

Euclid Prop. I, 16. In any triangle, if one of the sides be produced, the exterior angle is greater than either of the interior and opposite angles.

To show:
$\angle CBD > \angle ACB$.



Proof Let M be the midpoint of BC. Draw AM and produce to E so that AM = ME. Draw BE. Now:

$\Delta AMC \cong \Delta EMB$ by SAS

so $\angle EBC = \angle ACB$.

But $\angle CBD > \angle CBE$, so $\angle CBD > \angle ACB$. QED.

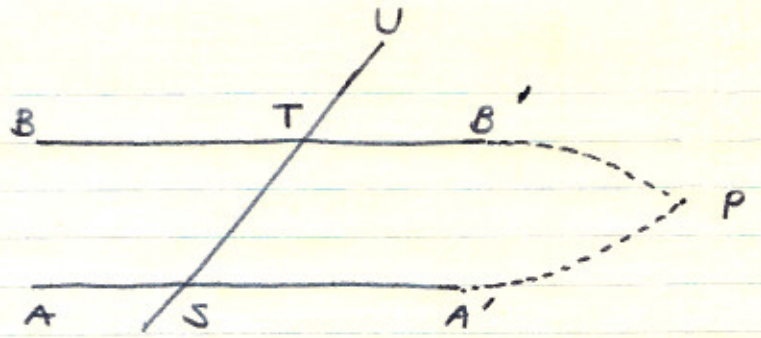
Note. This proof fails if lines are not infinite. Eg. E could lie on AD or on the opposite side of AD from M.

Euclid Prop. I, 17. In any triangle, two angles taken together in any manner are less than two right angles.

Proof In the figure above:

$\angle ABC + \angle ACB < \angle ABC + \angle CBD = 2\rho$. QED.

4/4 Euclid. Prop. I, 27. If a straight line falling on two straight lines make the alternate interior angles equal to one another, the straight lines will be parallel.



Given: $\angle AST = \angle B'TS$
To show: ~~$AA' \parallel BB'$~~.
AA' and BB' are parallel.

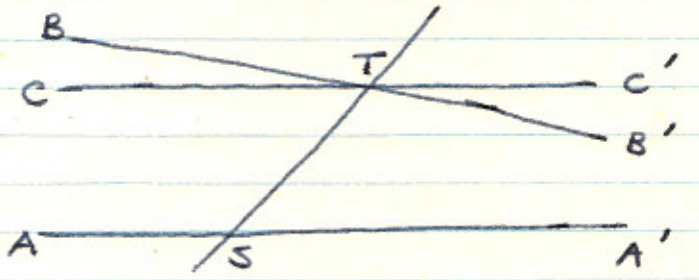
Proof Suppose AA' and BB' meet at P in the direction of A' and B'. In $\triangle PST$:
$\angle AST$ (exterior angle) $= \angle STP$ (interior angle) contradicting I, 16. Similarly for P on the other side of ST. QED

Remark Euclid Prop. I, 28 asserts we can conclude AA' ~~[illegible]~~ and BB' are parallel if:
i) either $\angle AST = \angle BTU$
ii) or $\angle AST + \angle BTS = 2\rho$.

Remark. The converse of I. 28 (Prop. I. 29) requires the Parallel Postulate. See Text - p. 18.

Deduction of Fifth Postulate from Playfair Axiom

Assume Playfair Axiom.



Given:
$\angle A'ST + \angle B'TS < 2\rho$.
To show: AA' and BB' meet in the direction of A' and B'.

Proof 1) Construct line CTC' through T so that $\angle A'ST + \angle C'TS = 2\rho$. Then ~~$CC' \parallel AA'$~~ by I. 28. Lines BB' and CC' are distinct, so by Playfair, BB' meets AA'. (CC' is parallel to AA')
2) To show: BB' meets AA' in the direction of A' and B'. Assume the contrary!

In $\triangle PST$:

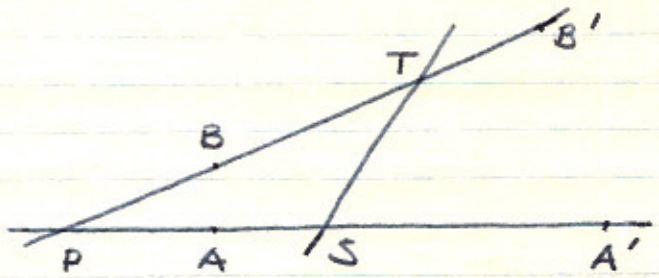
$\angle PST + \angle PTS > 2\rho$

by:

$\angle A'ST + \angle B'TS < 2\rho$

$\angle A'ST + \angle PST = 2\rho$

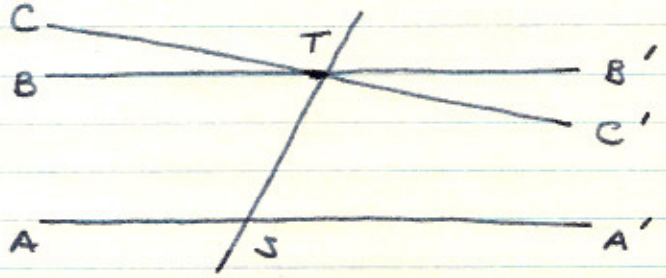
$\angle B'TS + \angle PTS = 2\rho$.



Contradicts I, 17. QED

Problem ② Deduce Playfair's Axiom from the Fifth Postulate. (Text. p. 21 #1).

Solution. Through T let a line meet AA' in S. Construct BB' through T so that $\angle A'ST + \angle B'TS = 2\rho$.



Then BB' ~~[illegible]~~ is parallel to AA' by I, 28. Suppose CC' is a second line through T parallel to AA'. If $\angle A'ST + \angle C'TS < 2\rho$, CC' would meet AA' by the Fifth Postulate. Similarly if $\angle A'ST + \angle C'TS > 2\rho$. Hence $\angle A'ST + \angle C'TS = 2\rho$. $\angle C'TS = \angle B'TS$ and the lines CC' and BB' coincide.

Some equivalents to the Fifth Postulate

1) The sum of the angles of a triangle equals two right angles
2) There exist triangles which are similar but not congruent.
3) There exist a pair of straight lines everywhere equally distant from one another.
4) Given any three points not lying in a straight line, there exists a circle passing through them.
5) If three of the angles of a quadrilateral are right angles, then so also is the fourth.
6) Through any point within an angle less than two-thirds of a right angle there can always be drawn a straight line which emits both sides of the angle.

Note. We will look at 1) and 2). See Text p. 25 ff for 3). See Sec 24 pp. 34 ff for 6).

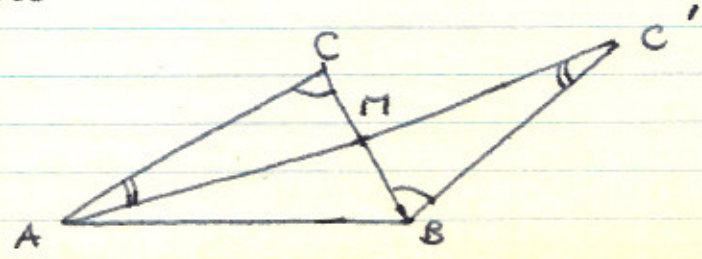
The angle-sum of a triangle

For $\triangle ABC$, let $\sigma_{ABC} = \angle A + \angle B + \angle C$ denote the sum of its angles.

Without assuming the Parallel Postulate we have:

Theorem (Saccheri-Legendre). The sum of the angles of a triangle is at most two right angles. i.e. $\sigma_{ABC} \leq 2\rho$.

Proof 1) Assume the contrary and $\triangle ABC$ has $\sigma_{ABC} = 2\rho + \varphi$, $\varphi > 0$.

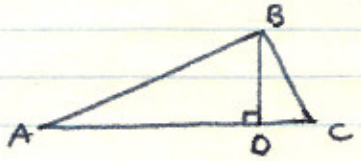


Let $\angle BAC = \alpha$ be the smallest angle in $\triangle ABC$. Let M be the midpoint of BC and produce AM to C' so $AM = MC'$. Then $\triangle AMC \cong \triangle C'MB$ by SAS so $\angle ACM = \angle C'BM$ and $\angle CAM = \angle BC'M$. Thus $\delta_{ABC'} = \delta_{ABC} = 2\rho + \varphi$. Since $\angle BAC' + \angle BC'A = \angle BAM + \angle MAC = \alpha$, one of these angles is $\leq \alpha/2$. Thus $\triangle ABC'$ is a triangle with the same angle sum $2\rho + \varphi$, and smallest angle $\leq \alpha/2$.

2) Repeat for all n times to obtain a triangle with angle sum $2\rho + \varphi$ and smallest angle $\leq \alpha/2^n$. Take n sufficiently large so that $\alpha/2^n < \varphi$. The sum of the larger two angles of this triangle must be $> 2\rho$. Contradiction to I, 17. QED.

Theorem If the angle-sum of some triangle is two right angles, then this is true for every triangle.

Proof. 1) Let $\triangle ABC$ have $\delta_{ABC} = 2\rho$. Then at least two angles, say $\angle A$ and $\angle C$, are acute. Locate D on line AC so $BD \perp AC$. D lies between A and C. Then



$$\delta_{ABC} = \delta_{ABD} + \delta_{DBC} - 2\rho = 2\rho$$

so

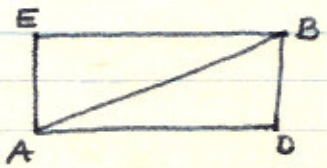
$$\delta_{ABD} + \delta_{DBC} = 4\rho.$$

But

$$\delta_{ABD} \leq 2\rho, \quad \delta_{DBC} \leq 2\rho$$

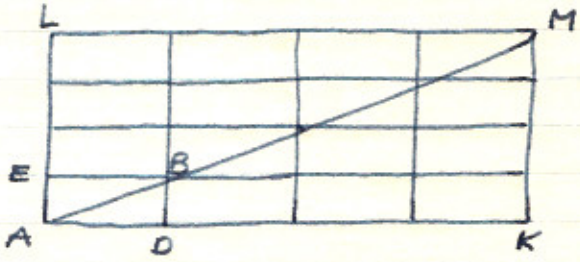
so equality must hold for both. i.e. Right triangle ABD has $\delta_{ABD} = 2\rho$.

2) Construct $\triangle ABE \cong \triangle ABD$ with $AD = ED$, $AE = DB$ and $\angle E = \rho$. Locate lines AE and BE first, use ASA.



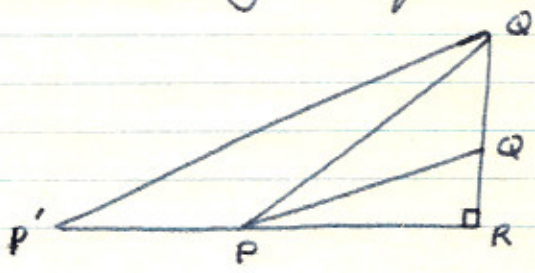
Then rect. ADBE has angle-sum $= 4\rho$.

4/7 3) Locate $D', D'', \dots$ on AD produced so that $AD = DD' = D'D'' = \dots$, n segments. Similarly n segments $AE = EE' = E'E'' = \dots$ on AE produced. The resulting n^2 rectangles are congruent to ADBE. (Proceed from ADBE through adjacent rectangles.)



Then rectangle AKML has angle-sum 4ρ. From $\delta_{AKM} + \delta_{ALM} = 4\rho$ and $\delta_{AKM} \le 2\rho$, $\delta_{ALM} \le 2\rho$, obtain $\delta_{AKM} = 2\rho$. ΔAKM is a rt. triangle with angle-sum 2ρ and legs arbitrarily large.

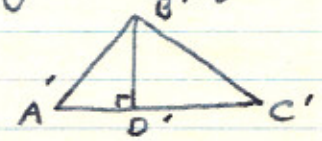
4) Let PQR be an arbitrary rt. triangle. On RP produced locate P', on RQ produced locate Q' so $P'R > PR$ and $Q'R > QR$ and right triangle $P'Q'R$ has $\delta_{P'Q'R} = 2\rho$. $\Delta P'Q'R$ is $\cong$ some ΔAMK above. Draw PQ'. Then



$$\delta_{P'Q'R} = \delta_{P'Q'P} + \delta_{PQ'R} - 2\rho = 2\rho$$

so $\delta_{P'Q'P} + \delta_{PQ'R} = 4\rho$ with summands $< 2\rho$. Hence $\delta_{PQ'R} = 2\rho$. Apply same argument to $\Delta PQ'R$ to get $\delta_{PQR} = 2\rho$. i.e. Every right triangle has angle-sum $= 2\rho$.

5) Let $A'B'C'$ be an arbitrary triangle. As in 1) locate D' so $B'D' \perp A'C'$ (assuming angles $\angle A', \angle C'$ acute).



Then $\delta_{A'B'C'} = \delta_{A'B'D'} + \delta_{D'B'C'} - 2\rho = 2\rho + 2\rho - 2\rho = 2\rho$.

Q.E.D.

Consequence Either the angle-sum of every triangle is $=2p$, or the angle sum of every triangle is $< 2p$.

Theorem a) The angle-sum of every triangle equals $2p$ is equivalent to:

Playfair's Axiom: Given a line l and a point P not lying on l, then there exists in the plane determined by l and P one and only one line which contains P but not any point of l.

b) The angle-sum of every triangle is less than $2p$ is equivalent to:

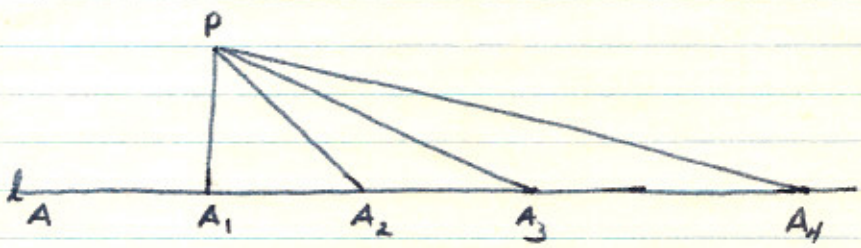
Non-Euclidean Axiom: Given a line l and a point P not lying on l, then there exists in the plane determined by l and P at least two lines which contain P but not any point of l.

Proof of: Angle-sum $= 2p$ implies Playfair Axiom.

1) Through a given point P not on a given line l there exist a line making with l an angle less than any prescribed angle not zero.

Note. Angle-sum $= 2p$ implies exterior angle of a triangle equals sum of opposite interior angles.

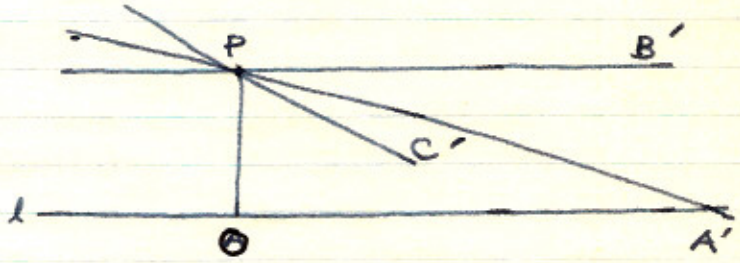
Locate A_1 on l so that $PA_1 \perp l$.
Locate A_2 on l so that $PA_1 = A_1A_2$.
Then



$\angle A_1A_2P = \angle A_1PA_2$ and sum is $\angle AA_1P = p$
so $\angle A_1A_2P = p/2$.
Locate A_3 on l so that $PA_2 = A_2A_3$.
Then
$\angle A_2A_3P = \angle A_2PA_3$ and sum is $\angle A_1A_2P = p/2$
so $\angle A_2A_3P = p/2^2$.

Repeat with $PA_3 = A_3A_4$, $PA_4 = A_4A_5$, ... After n steps obtain $\angle A_nA_{n+1}P = p/2^n$.

2) Let $P\bigcirc \perp l$ and $PB' \perp P\bigcirc$. Then PB' and l are parallel by I.28. Let PC' be any other line through P. Say PC' is interior to angle $\bigcirc PB'$. Then $\angle C'PB' > 0$. Locate A' on l so that $\angle PA'\bigcirc < \angle C'PB'$. Then



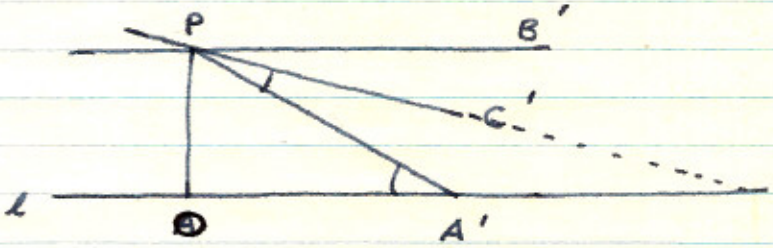
$\angle A'PB' + \angle \bigcirc PA' = p$ so $\angle PA'\bigcirc = \angle A'PB'$

$\angle \bigcirc PA' + \angle PA'\bigcirc + \underbrace{\angle A'\bigcirc P}_{p} = 2p$

Then $\angle A'PB < \angle C'PB'$ and PC' meets l (essentially by Pasch's Axiom). QED.

Proof of: Angle-sum $< 2p$ implies NE Axiom.

Let $P\bigcirc \perp l$ and $PB' \perp P\bigcirc$. Then PB' and l are parallel by I.28. Let $A' \neq \bigcirc$ be arbitrary. Let line PC' here be so that $\angle \bigcirc A'P = \angle A'PC'$.



1) PC' does not meet l. For if PC' meets l, the resulting triangle would have an exterior angle equal to an interior angle.

2) PC' and PB' are distinct lines. For:

$\angle \bigcirc PA' + \angle PA'\bigcirc + \underbrace{\angle A'\bigcirc P}_{p} < 2p$

so

$\angle A'PC' + \angle \bigcirc PA' < p$

but $\angle A'PB' + \angle \bigcirc PA' = p$

so $\angle A'PC' < \angle A'PB'$. QED

Remark Euclid I.27 or I.28 shows at least one line parallel to a given line through a given point exist. We have shown:

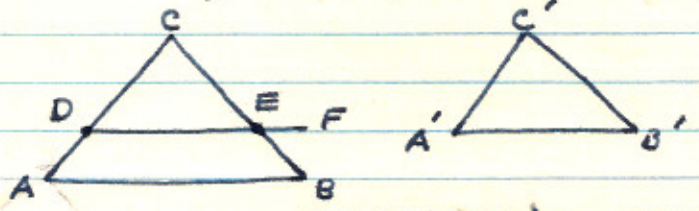
$$\begin{cases} \text{angle-sum} = 2\rho \\ \text{angle-sum} < 2\rho \end{cases} \text{ implies } \begin{cases} \text{Playfair Axiom (at most one)} \\ \text{NE Axiom (at least two)} \end{cases}$$

The assertions on both sides of the implication are exclusive and exhaust all possibilities. Hence they are respectively equivalent. QED for theorem.

The existence of similar triangles

Theorem The existence of two similar but not congruent triangles is equivalent to Playfair's Axiom.

Proof 1) Playfair's Axiom implies the existence of similar triangles is consequence of elementary Euclidean geometry.



2) Given:
$\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$.
$A'B' < AB$.
Show: There exists a triangle with angle sum 2ρ.
Locate D on AC so $DC = A'C'$. Let line DF have $\angle CDF = \angle C'A'B'$. DF meets AB or CB; it cannot meet AB by I.28 and $\angle CDF = \angle CAB$, so DF meets CB, say in E. Then $\triangle DEC \cong \triangle A'B'C'$ by ASA.

3) Since $\measuredangle_{ABC} = \measuredangle_{DEC}$, quadrilateral ABED has angle-sum 4ρ. Draw BD. Then

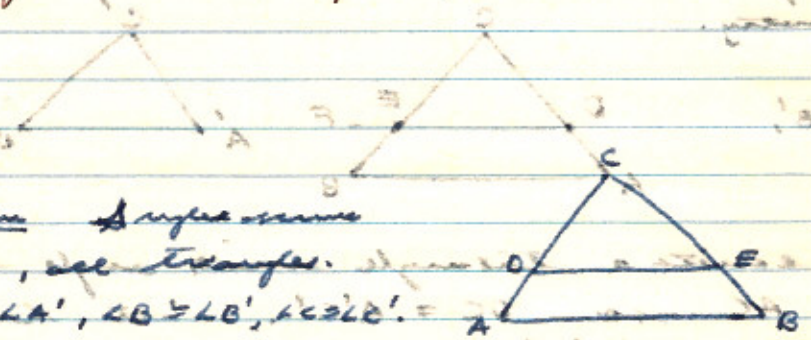


$\measuredangle_{ABD} + \measuredangle_{BDE} = 4\rho$.
and each summand is $< 2\rho$, so each equals 2ρ.
i.e. $\triangle ABD$ has $\measuredangle_{ABD} = 2\rho$. QED

Note: Here may be pointed: If the NE Axiom holds, similar triangles are congruent. See Theorem p. P3.

(OVER) →

Proof Suppose false and $A'C' < AC$, say. On AC let $DC = A'C'$ and on BC let $EC = B'C'$. Then $\triangle DEC \cong \triangle A'B'C'$ by SAS. Then $\angle CDE = \angle C'A'B' = \angle CAB$. Hence E lies between B and C by I.28. $ABED$ is a quadrilateral with exterior angle equal to interior opposite angle. Contradiction. QED.



Given Angle-angle-angle, two triangles. $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$.

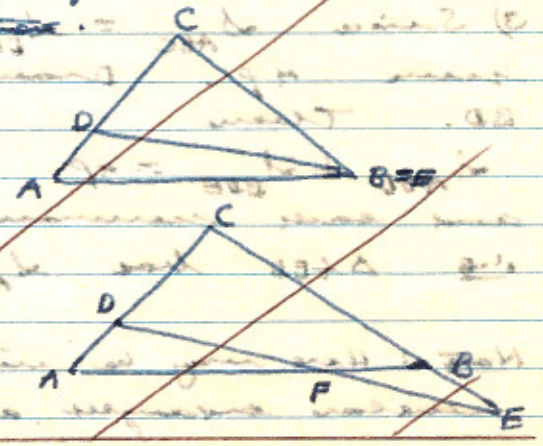
Theorem $\triangle ABC \cong \triangle A'B'C'$.

Proof Suppose false. Say $A'C' < AC$. Let $DC = A'C'$, $EC = B'C'$. Then $\triangle DEC \cong \triangle A'B'C'$ by SAS.

1) If $EC < BC$ $\triangle ADE$ has exterior angle equal to an opposite interior angle. Contradiction.

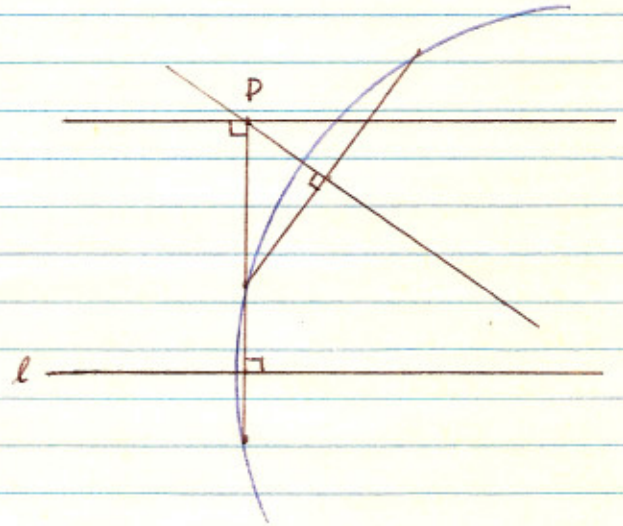
2) If $EC = BC$, $\triangle ADB$ has exterior angle equal to an opposite interior angle. Contradiction.

3) If $EC > BC$, $\triangle AFD$ has exterior angle at D equal to interior angle at A. Contradiction. QED.



Problems ③ Show that the assertion "A circle may be passed through any three points not lying on a line." implies Playfair's Axiom. (W. Bolyai). Note: Converse is contained in elementary Euclidean geometry.

Hint:



Ref. M.J. Greenberg. Euclidean and Non-Euclidean Geometries. p. 135.

4/9 3. Historical sketch.

Read: Text. Chs. II and III.
Especially §17 to 35.