Lie frame

Euclidean case $\qquad L = \lambda_0 \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1_n & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$

$E = (E_0\ E_1 \cdots E_n\ E_r\ E_s)$

$(E_i | E_j) = ij^{\text{th}}$ entry of L $\qquad$ $(\xi|\eta) = {}^t\xi L \eta$

${}^tE_i \cdot L E_j = ij^{\text{th}}$ entry of L

or $\quad {}^tELE = L.$

$O = O(n)$ =: space of Lie frames

Λ^{2n-1} = space of lines in Ω^{n+1}

$\Lambda = O / O_{\langle \alpha, \beta \rangle}$ $\qquad$ homogeneous space.

$O_{\langle \alpha, \beta \rangle}$ = stability subgroup, $\quad \langle \alpha, \beta \rangle \subset \Omega$

$= \{ g \in O \mid g \langle \alpha, \beta \rangle \subset \langle \alpha, \beta \rangle \}$

$dE_j = \sum_\alpha E_\alpha \omega^\alpha_j$ $\qquad$ moving Lie frame.

$dE = E\omega$ $\qquad$ Maurer-Cartan form.

$\omega = E^{-1} dE$ $\qquad$ Chern (2.6)

ω is left invariant on $GL(n+3, \mathbb{R})$!

$$L_g^* \omega = (gE)^{-1} d(gE) = E^{-1} g^{-1} g\, dE = E^{-1} dE = \omega.$$

Also: $\quad d^2E = d(E\omega) = dE \wedge \omega + E\, d\omega$

$= E\omega \wedge \omega + E\, d\omega = E(\omega \wedge \omega + d\omega)$

$= 0$, so $\qquad d\omega + \omega \wedge \omega = 0$ $\qquad$ $d(f\alpha) = df \wedge \alpha + f\, d\alpha.$

Maurer-Cartan equation. Chern (2.10).

Differentiate $\,^tELE = L$:

$$\,^t(dE)LE + \,^tELdE = 0$$

$$\,^t\omega\,^tELE + \,^tELE\omega = 0$$

$$\,^t\omega L + L\omega = 0 \quad \text{or} \quad \,^t(L\omega) + (L\omega) = 0$$

i.e. ω has values in the Lie algebra

$$\underline{o} = \{ X \in \mathbb{R}^{(n+3)\times(n+3)} \mid \,^tXL + LX = 0 \}$$

of O.

Contact density

Fix $\alpha, \beta \in \mathbb{R}^{n+3}$ so $\langle \alpha, \beta \rangle \subset \Omega^{n+1}$

Consider the 1-form on O :

$$\,^t\alpha\,(E\,l\,dE)\beta = \,^t\alpha\,^tELdE\beta = \,^t\alpha\,^tELE\omega\beta$$

$$= \,^t\alpha(L\omega)\beta .$$

If $g \in O_{\langle\alpha,\beta\rangle}$, then $g(\alpha,\beta) = (\alpha,\beta)\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

or $\quad g\alpha = \alpha a + \beta b$
$\quad\;\; g\beta = \alpha c + \beta d$

so

$$R_g^*\left(\,^t\alpha(E\,l\,dE)\beta\right) = \,^t\alpha\,^t(Eg)L\,d(Eg)\beta$$

$$= \,^t(g\alpha)\,^tELdE(g\beta) = \,^t(g\alpha)L\omega(g\beta)$$

$$= \,^t(\alpha a+\beta b)L\omega(\alpha c+\beta d) = (ad-bc)\,^t\alpha(L\omega)\beta$$

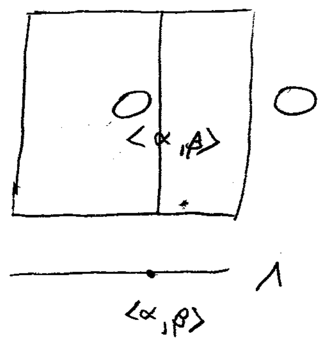
since $L\omega$ is 2-dimensional skew.

Hence : $\,^t\alpha(E\,l\,dE)\beta = \,^t\alpha L\omega\beta$

is right invariant up to scalar factors

Likewise: ${}^t\alpha L\omega\beta$ is defined by $\langle\alpha,\beta\rangle$ up to a scalar since a basis change of $\langle\alpha,\beta\rangle \subset \Omega^{n+1}$ is $(\alpha\ \beta) \rightsquigarrow (\alpha\ \beta)\begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Thus:



${}^t\alpha L\omega\beta$ is:

i) left invariant under O

ii) right invariant up to scalars under $O_{\langle\alpha,\beta\rangle}$

${}^t\alpha(E|dE)\beta = {}^t\alpha L\omega\beta$ - contact form on $\Lambda^{2n-1} = O/O_{\langle\alpha,\beta\rangle}$ is well-defined up to scalars.

<u>Special case</u>

$$\alpha = e_0 = \begin{pmatrix} 1 \\ \hline 0 \\ \vdots \\ 0 \\ \hline 0 \\ \hline 0 \end{pmatrix}, \quad \beta = e_n - e_r = \begin{pmatrix} 0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline 1 \\ \hline -1 \\ 0 \end{pmatrix}$$

The Transpose

$${}^te_0(dE|E)(e_n - e_r)$$

is used by Chern - after change of coordinates.

$$\xi = \begin{pmatrix} 1 \\ \hline x_1 \\ \vdots \\ x_n \\ \hline 0 \\ xx \end{pmatrix}, \quad \pi = \begin{pmatrix} 0 \\ \hline -p_1 \\ \vdots \\ -p_{n-1} \\ 1 \\ \hline \cancel{-px+1} \\ 2(x_n - px) \end{pmatrix}$$

Find $J : \mathbb{R}^n \times \mathbb{R}^{n-1} \longrightarrow O$

such that

$$\left\{\begin{array}{l} J(x,p)\, e_0 = \xi \\ J(x,p)\, (e_n - e_r) = \pi \end{array}\right\} \quad E = J(x,p)$$

Then

$${}^t(e_n - e_r)\,(E \mid dE)\, e_0 \Big|_{E = J(x,p)} = {}^t(e_n - e_r)\, {}^tJ(x,p)\, L\, dJ(x,p)\, e_0$$

$$= {}^t(J(x,p)(e_n - e_r))\, L\, (d(J(x,p)\, e_0))$$

$$= {}^t\pi\, L\, d\xi = (\pi \mid d\xi)$$

contact form (up to scalar) — on Λ^{2n-1}.

should obtain (const). $(dx_n - p_1\, dx_1 - \cdots - p_{n-1}\, dx_{n-1})$.