The group of Lie geometry

Let $O = O(W)$ be the orthogonal group of $W = e_0 h + V + e_r h + e_s h$ with the non-singular quadratic form $(\mathfrak{z}|\mathfrak{z}) = \lambda_0(\bar{\mathfrak{z}}\cdot\bar{\mathfrak{z}} - (\mathfrak{z}^r)^2 - \mathfrak{z}^0\mathfrak{z}^{-0})$.

By Witt's theorem, O is generated by reflections in non-singular subspaces of co-dimension one!

$$\mathfrak{z} \rightsquigarrow \mathfrak{z} - \beta\,\frac{2(\beta|\mathfrak{z})}{(\beta|\beta)}, \qquad \beta \in W,\ (\beta|\beta) \neq 0 .$$

Aside: O is also generated by such reflections for which $(\beta|e_s) \neq 0$.

The group G of Lie geometry is the group of transformations on Lie cycles which is generated by Lie contact inversions i.e. by

$$\langle\sigma\rangle \rightsquigarrow \langle\sigma - \beta\,\frac{2(\beta|\sigma)}{(\beta|\beta)}\rangle, \quad \langle\beta\rangle \in \mathbb{P}^{n+2} \setminus (\Omega^{n+1} \cup \langle e_s\rangle^{\perp}).$$

Clearly: $G = PO(W) = O/h^*$,

where h^* is the center of O consisting of $\mathfrak{z} \rightsquigarrow \mathfrak{z}a$, $a \neq 0$ in h.

Matrix representation

$$W = \mathbb{R}^{n+3} \text{ (col)}, \qquad \zeta = \begin{pmatrix} \zeta^0 \\ \hline \zeta^1 \\ \vdots \\ \zeta^n \\ \hline \zeta^r \\ \zeta^s \end{pmatrix}$$

Bilinear form

$$(\zeta | \eta) = {}^t\zeta L \eta$$

with

$$L = \lambda_0 \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & A & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$

Gram matrix for Lie geometry

From

$$\zeta - \beta \frac{2(\beta|\zeta)}{(\beta|\beta)} = \left(1_{n+3} - 2 \frac{\beta \, {}^t\beta L}{{}^t\beta L \beta} \right) \zeta ,$$

we have thus:

O is generated by $1_{n+3} - 2 \dfrac{\beta \, {}^t\beta L}{{}^t\beta L \beta}$

with ${}^t\beta L \beta \neq 0$.

<u>The Lie quadric as homogeneous space</u>

$$\left.\begin{aligned} O &= O(W) \\ G &= O/\mathbb{R}^* \end{aligned}\right\} \text{ are transitive on } \Omega^{n+1} \text{ by Witt's theorem.}$$

$$e_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \langle e_0\rangle^{\perp} = \langle e_0\rangle + V + \langle e_r\rangle = \left\{\begin{pmatrix} * \\ * \\ \vdots \\ * \\ * \\ 0 \end{pmatrix}\right\}$$
$$= \{ \xi^{r} = 0 \}$$

Let $O_{\langle e_0\rangle}$ = stability subgroup of $\langle e_0\rangle \subset W$.

For $g \in O_{\langle e_0\rangle}$ we have

$$\left.\begin{aligned} g\langle e_0\rangle &\subset \langle e_0\rangle \\ g\langle e_0\rangle^{\perp} &\subset \langle e_0\rangle^{\perp} \end{aligned}\right\}, \text{ so } g = \left(\begin{array}{c|c|cc} * & * & * & * \\ \hline 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{array}\right).$$

Thus:
$$\Omega^{n+1} = O/O_{\langle e_0\rangle} = (O/\mathbb{R}^*)/(O_{\langle e_0\rangle}/\mathbb{R}^*) = G/G_{\langle e_0\rangle}.$$

<u>Note</u> $G_{\langle e_0\rangle}$ is a parabolic subgroup of $G = PO(W)$.

$$\cancel{\dim O = \frac{(n+1)4}{2}}$$
$$\cancel{\dim O_{\langle e_0\rangle} = ?}$$

The space of contact elements

A <u>contact element</u> is a (parabolic) pencil of Lie cycles which are in oriented contact i.e. a (projective) line lying in Ω^{n+1} or a totally isotropic 2-dimensional subspace of W.

By Witt's theorem, $O = O(W)$ and $G = O/h^*$ are transitive on contact elements.

$$\Lambda = \begin{pmatrix} \text{space of} \\ \text{contact} \\ \text{elements} \end{pmatrix} = \begin{pmatrix} \text{homogeneous} \\ \text{space of } G \text{ or } O \end{pmatrix}$$

Let

$$\xi = \left(\begin{array}{c} 1 \\ \hline x_1 \\ \vdots \\ x_{n-1} \\ x_n \\ \hline 0 \\ xx \end{array}\right) \quad \text{and} \quad \pi = \left(\begin{array}{c} 0 \\ \hline -p_1 \\ \vdots \\ -p_{n-1} \\ 1 \\ \hline -\sqrt{pp+1} \\ 2(x_n - px) \end{array}\right) \leftarrow \text{why "$-$" sign?}$$

where

$$xx = \sum_1^n x_i^2, \quad px = \sum_1^{n-1} p_i x_i, \quad pp = \sum_1^{n-1} p_i^2$$

Easy to check:

$$(\xi | \xi) = 0, \quad (\xi | \pi) = 0, \quad (\xi | \pi) = 0$$

so $\langle \xi, \pi \rangle \subset \Omega^{n+1}$ is a contact element.

$$\mathfrak{z} + \pi\tau = \begin{pmatrix} 1 \\ x_1 - p_1\tau \\ \vdots \\ x_{n-1} - p_{n-1}\tau \\ x_n + \tau \\ \hline -\sqrt{pp+1}\,\tau \\ xx - 2(x_n - px)\tau \end{pmatrix}$$

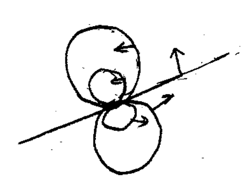
is the pencil of cycles:

$$(x_1' - x_1 + p_1\tau)^2 + \cdots + (x_{n-1}' - x_{n-1} + p_{n-1}\tau)^2$$
$$+ (x_n' - x_n - \tau)^2 = (pp+1)^2\tau^2$$

having oriented contact with the hyperplane $(\tau = \infty)$

$$(x_1' - x_1)p_1 + \cdots + (x_{n-1}' - x_{n-1})p_{n-1} - (x_n' - x_n) = 0$$

which passes through the point $(x_1, \dots, x_n)$ $(\tau = 0)$ with oriented "normal" $(-p_1, \dots, -p_{n-1}, 1)$. Here $x_1', \dots, x_n'$ are "running coordinates".



Special case:

$x_1 = 0, \dots, x_{n-1} = 0, x_n = 0$

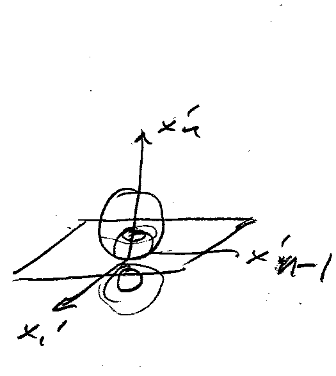
$p_1 = 0, \dots, p_{n-1} = 0$.

Then

$$\langle \mathfrak{z}, \pi \rangle \text{ is } l_0 = \left\langle \begin{pmatrix} 1 \\ \hline 0 \\ \vdots \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \hline 0 \\ \vdots \\ 0 \\ 1 \\ \hline -1 \\ 0 \end{pmatrix} \right\rangle$$

which is the pencil

$$x_1'^2 + \cdots + x_{n-1}'^2 + (x_n' - \tau)^2 = \tau^2$$



One has

$$\Lambda = G/G_{\ell_0} = O/O_{\ell_0}.$$

By computing their Lie algebras, one has

$$\dim G = \tfrac{1}{2}(n+3)(n+2)$$

$$\dim G_{\ell_0} = \tfrac{1}{2}(n+3)(n+2) - 2n+1$$

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so $\dim \Lambda = 2n-1.$

Note

$$\xi(x,p) = \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_{n-1} \\ x_n \\ 0 \\ xx \end{pmatrix}, \qquad \pi(x,p) = \begin{pmatrix} 0 \\ -p_1 \\ \vdots \\ -p_{n-1} \\ 1 \\ -\sqrt{1+pp} \\ 2(x_n - px) \end{pmatrix}$$

are functions of $x_1, \dots, x_n, p_1, \dots, p_{n-1}$, and

$$\mathbb{R}^n \times \mathbb{R}^{n-1} \longrightarrow \Lambda^{2n-1}$$

$$(x, p) \longmapsto \langle \xi(x,p), \pi(x,p) \rangle$$

has an open dense image in Λ^{2n-1}.