Projective setting

Introduce a projective space to carry the redundant coordinates $\alpha^0, \dots, \alpha^\infty$:

$$W = e_0 h + V + e_r h + e_\infty h$$

vector space of dimension $n+3$, whose typical element is

$$\mathfrak{z} = e_0 \mathfrak{z}^0 + \vec{\mathfrak{z}} + e_r \mathfrak{z}^r + e_\infty \mathfrak{z}^\infty \qquad \mathfrak{z}^0, \mathfrak{z}^r, \mathfrak{z}^\infty \in h, \quad \vec{\mathfrak{z}} \in V$$

Aside: For $V = R^n$ (coe) we write:

$$\mathfrak{z} = \begin{pmatrix} \mathfrak{z}^0 \\ \hline \mathfrak{z}^1 \\ \vdots \\ \mathfrak{z}^n \\ \hline \mathfrak{z}^r \\ \mathfrak{z}^\infty \end{pmatrix}.$$

The projective space $\mathbb{P}^{n+2} = P(W)$ consists of all one-dimensional subspaces $\langle \mathfrak{z} \rangle$ of W, where $\langle \cdots \rangle$ denotes "span".

"Extend" the form on V to W by:

$$(\mathfrak{z} | \mathfrak{z}) = \lambda_0 (\vec{\mathfrak{z}} \cdot \vec{\mathfrak{z}} - (\mathfrak{z}^r)^2 - \mathfrak{z}^0 \mathfrak{z}^\infty),$$

where $\lambda_0 \neq 0$ is an arbitrary but fixed scalar of h.

For
$$\mathfrak{z} = e_0 \mathfrak{z}^0 + \vec{\mathfrak{z}} + e_r \mathfrak{z}^r + e_\infty \mathfrak{z}^\infty$$
$$\eta = e_0 \eta^0 + \vec{\eta} + e_r \eta^r + e_\infty \eta^\infty$$
we have

$$(\mathfrak{z} | \eta) = \lambda_0 (\vec{\mathfrak{z}} \cdot \vec{\eta} - \mathfrak{z}^r \eta^r - \tfrac{1}{2} \mathfrak{z}^0 \eta^\infty - \tfrac{1}{2} \mathfrak{z}^\infty \eta^0),$$

so $(-|-)$ has Gram matrix

$$\lambda_0 \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & A & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$

with respect to basis $e_0, e_1 \cdots e_n, e_r, e_\infty$,

where A is the Gram matrix of $\vec{x}\cdot\vec{x}$ with respect to a basis $e_1, \dots, e_n$ of V. The quadratic form $(-|-)$ is non-singular.

The Lie quadric $\Omega^{n+1} \subset \mathbb{P}^{n+2}$ is the non-singular quadric

$$\Omega^{n+1} = \{ \langle \mathfrak{z} \rangle \in \mathbb{P}^{n+1} \mid (\mathfrak{z}|\mathfrak{z}) = 0 \}$$

A point $\langle \alpha \rangle \in \Omega^{n+1}$ will be called an oriented sphere or Lie cycle.

Note: If $\vec{x}\cdot\vec{x}$ has signature $(\underbrace{- \cdots -}_{s}\ \underbrace{+ \cdots +}_{n-s})$ then $(-|-)$ has signature $(s+2, n-s+1)$. Especially for $V = \mathbb{R}^n$ Euclidean space, Ω^{n+1} has signature $(2, n+1)$, and contains lines but no higher dimensional flats.

<u>Polars</u> a) Pole and polar with respect to Ω^{n+1} are

$$\langle \zeta \rangle \text{ and } \langle \zeta \rangle^{\perp} = \{ \langle \xi \rangle \in \mathbb{P}^{n+2} \mid (\zeta | \xi) = 0 \}.$$

b) The <u>tangent prime</u> to Ω at a point $\langle \alpha \rangle \in \Omega$ is $\langle \alpha \rangle^{\perp}$. For such a tangent prime, $\langle \alpha \rangle^{\perp} \cap \Omega$ is a <u>projective cone</u> (singular as a quadric).

c) Note especially:

$$\langle e_0 \rangle^{\perp} = \langle e_0 \rangle + V + \langle e_r \rangle$$

$$\langle e_r \rangle^{\perp} = \langle e_0 \rangle + V + \langle e_s \rangle$$

since $(e_0 | e_0) = 0$, $(e_0 | e_s) = -\frac{1}{2}$ and etc.

<u>Some projective extensions of familiar spaces</u>

1) Original space V

$$V \xrightarrow{incl} \langle e_0 \rangle^{\perp} \cap \langle e_r \rangle^{\perp} = \langle e_0 \rangle + V$$

$$\vec{x} \rightsquigarrow \langle e_0 + \vec{x} \rangle. \quad \text{inclusion}$$

2) Original space and additional "z-"axis

$$V \times \mathbb{R} \xrightarrow{incl} \langle e_r \rangle^{\perp}$$

$$(\vec{x}, z) \rightsquigarrow \langle e_0 + \vec{x} + e_s z \rangle$$

3) Center and signed radius

$$V \times \mathbb{R} \xrightarrow{incl} \langle e_0 \rangle^{\perp}$$

$$(\vec{a}, a^r) \rightsquigarrow \langle e_0 + \vec{a} + e_r a^r \rangle$$