Invariants of Lie geometry

Classical cross-ratio — on a line with coordinates

$$(LM:AB) = \frac{LA}{AM} \Big/ \frac{LB}{BM} \quad \text{or} \quad \frac{a-l}{m-a} \Big/ \frac{b-l}{m-b}.$$

$$\frac{AL}{LB} \Big/ \frac{AM}{MB} = \frac{AL}{LB}\,\frac{MB}{AM}$$

Quadratic form on

$$W = e_0 h + V + e_r h + e_\infty h$$

is

$$(\zeta|\eta) = \lambda_0 \left(\vec{\zeta}\cdot\vec{\eta} - \zeta^r \eta^r - \tfrac{1}{2}(\zeta^0 \eta^\infty + \zeta^\infty \eta^0)\right).$$

Lie quadric: in $\mathbb{P}^{n+2} = \mathbb{P}(W)$!

$$\Omega^{n+1} = \{ \langle \zeta \rangle \in \mathbb{P}^{n+2} \mid (\zeta|\zeta) = 0 \}$$

Four point invariant:

$$F(\langle\lambda\rangle, \langle\mu\rangle : \langle\alpha\rangle, \langle\beta\rangle) = \frac{(\lambda|\alpha)}{(\alpha|\mu)} \Big/ \frac{(\lambda|\beta)}{(\beta|\mu)}$$

– clear well defined on $\mathbb{P}^{n+2}$

General bunch

choose $\langle\lambda\rangle, \langle\mu\rangle \in \mathbb{P}^{n+2}$

$\langle\alpha\rangle \in \Omega$ central cycle

$a \in h$ frames

$$\text{bunch} = \{ \langle\zeta\rangle \in \Omega \mid F(\langle\lambda\rangle, \langle\alpha\rangle : \langle\mu\rangle, \langle\zeta\rangle) = a \}$$

$$\frac{(\lambda|\mu)}{(\mu|\alpha)} \Big/ \frac{(\lambda|\xi)}{(\xi|\alpha)} = a \quad \text{or} \quad \frac{(\lambda|\mu)(\alpha|\xi)}{(\alpha|\mu)(\lambda|\xi)} = a$$

$$(\alpha(\lambda|\mu) - \lambda(\alpha|\mu)a \mid \xi) = 0$$

The bunch is

$$\langle \alpha(\lambda|\mu) - \lambda(\alpha|\mu)a \rangle^{\perp} \cap \Omega^{n+1}.$$

Note! Need: $\alpha(\lambda|\mu) - \lambda(\alpha|\mu)a \neq 0$ in W.

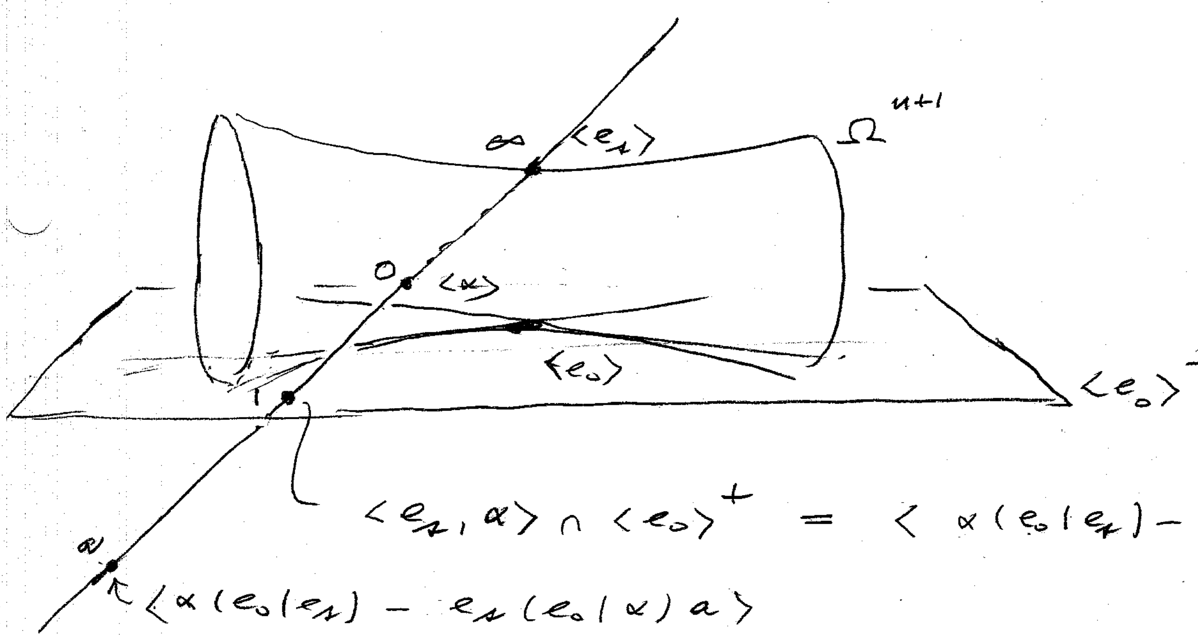
Bunches — ________ ruled?

$$F\left(\langle e_1\rangle, \langle \alpha\rangle : \langle e_0\rangle, \langle \zeta\rangle\right) = a \qquad \begin{cases} \langle\alpha\rangle \in \Omega \\ a \in h \end{cases}$$

$$\frac{(e_1|e_0)}{(e_0|\alpha)} \Big/ \frac{(e_1|\zeta)}{(\zeta|\alpha)} = a$$

or
$$\frac{(e_0|e_1)(\alpha|\zeta)}{(e_0|\alpha)(e_1|\zeta)} = a$$

$$\left(\alpha(e_0|e_1) - e_1(e_0|\alpha)a \;\middle|\; \zeta\right) = 0$$



$$\langle e_1, \alpha\rangle \cap \langle e_0\rangle^{\perp} = \langle \alpha(e_0|e_1) - e_1(e_0|\alpha)\rangle$$

$$\langle \alpha(e_0|e_1) - e_1(e_0|\alpha)a\rangle$$

Classical cross-ratio of points on $\langle e_1, \alpha\rangle$:

$$\left(\langle e_1\rangle, \langle\alpha\rangle : \langle e_1,\alpha\rangle \cap \langle e_0\rangle^{\perp}, \langle \alpha(e_0|e_1) - e_1(e_0|\alpha)a\rangle\right)$$

$$= (\infty\ 0 ; 1\ a) = \frac{1-\infty}{0-1} \Big/ \frac{a-\infty}{0-a}$$

$$= a$$

Fix $\langle e_0\rangle, \langle e_\infty\rangle \in \Omega^{n+1}$

~~not~~ tangent! $(e_0 | e_\infty) \neq 0$.

Then have bijection

$$\langle\beta\rangle \rightsquigarrow \begin{cases} \langle\alpha\rangle = \text{second intersection of } \langle e_\infty, \beta\rangle \text{ and } \Omega \\ \qquad = \langle e_\infty(\beta|\beta) - \beta\, 2(e_\infty|\beta)\rangle \\ a = (\langle e_\infty\rangle, \langle\alpha\rangle : \langle e_\infty, \beta\rangle \cap \langle e_0\rangle^{\perp}, \langle\beta\rangle) \\ \text{classical cross ratio.} \end{cases}$$

pair $\langle\alpha\rangle, a \rightsquigarrow \langle \alpha(e_0|e_\infty) - e_\infty(e_0|\alpha)a\rangle$

of

$$\mathbb{P}^{n+2} \setminus (\Omega^{n+1} \cup \langle e_\infty\rangle^{\perp}) \xrightarrow{?} (\Omega^{n+1} \setminus \langle e_\infty\rangle) \times (k \cup \{\infty\})$$

$\boxed{\text{Needs to be checked}}$ - compute compositions of maps in both orders.

$\langle e_\infty + \beta\tau\rangle \in \Omega$ for $(e_\infty + \beta\tau | e_\infty + \beta\tau) = 0$

or $0 + 2(e_\infty|\beta)\tau + (\beta|\beta)\tau^2 = 0$; so $\tau = -2(e_\infty|\beta)/(\beta|\beta)$

and point is $\langle e_\infty(\beta|\beta) - \beta\, 2(e_\infty|\beta)\rangle$

Types of bunches

$$\langle\beta\rangle^{\perp} \cap \Omega^{n+1} \quad , \quad \langle\beta\rangle \in \mathbb{P}^{n+2} \setminus \Omega^{n+1}$$

$$(\xi|\eta) = \lambda_0 \left(\vec{\xi}\cdot\vec{\eta} - \xi^r\eta^r - \tfrac{1}{2}(\xi^0\eta^\infty + \xi^\infty\eta^0) \right)$$

$$\Omega^{+} = \{ \langle\xi\rangle \in \mathbb{P}^{n+2} \mid (\xi|\xi)/\lambda_0 > 0 \}$$

$$\Omega^{-} = \{ \text{———} \mid \text{———} < 0 \}$$

$$\mathbb{P}^{n+2} = \Omega^{n+1} \cup \Omega^{+} \cup \Omega^{-} \quad \text{disjoint union}$$

Suppose V of $\vec{\xi}\cdot\vec{\eta}$ has signature

$$(\underbrace{-----}_{s}\ \underbrace{++-+}_{n-s})$$

$$W = \underset{-\ +}{\langle e_0, e_\infty\rangle} + \underset{(s,n-s)}{V} + \underset{-}{\langle e_r\rangle}$$

has signature $(s+2, n-s+1)$.

$$W = \langle\beta\rangle + \langle\beta\rangle^{\perp}$$

i) $\langle\beta\rangle \in \Omega^{+}$ $\qquad W = \underset{(0,1)}{\langle\beta\rangle} + \underset{(s+2,\, n-s)}{\langle\beta\rangle^{\perp}}$

so $\langle\beta\rangle^{\perp} \cap \Omega^{n+1}$ has signature $(s+2, n-s)$

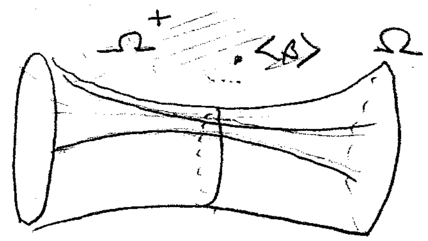
ii) $\langle\beta\rangle \in \Omega^{-}$ $\qquad$ ———

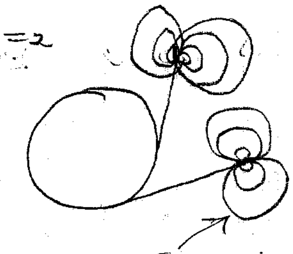
so $\langle\beta\rangle^{\perp} \cap \Omega^{n+1}$ has signature $(s+1, n-s+1)$.

Case V = Euclidean space $t=0$

i) $\langle\beta\rangle \in \Omega^+$, ruled bunch.

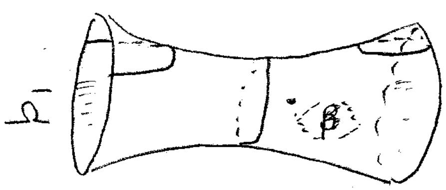
$\langle\beta\rangle^\perp \cap \Omega$ has signature $(2, n)$





ii) $\langle\beta\rangle \in \Omega^-$, unruled bunch

$\langle\beta\rangle^\perp \cap \Omega$ has signature $(1, n+1)$



$n=2$

EXAMPLE ~~NEEDED~~.

Eg $\quad \langle\beta\rangle = \langle \alpha (e_0|e_s) - e_s (e_0|\alpha) a \rangle$, $\quad \langle\alpha\rangle \in \Omega$

$$(\beta|\beta) = (\text{const})^2 \left(\cancel{(\alpha|\alpha)}(e_0|e_s)^2 - 2a(e_0|e_s)(e_0|\alpha)(\alpha|e_s) \right.$$

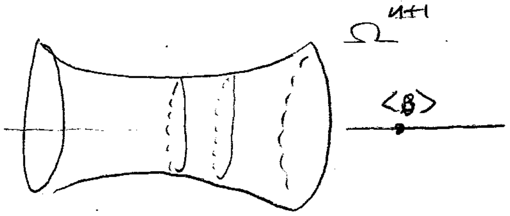
$$\left. + \cancel{(e_s|e_s)}(e_0|\alpha)^2 a^2 \right)$$

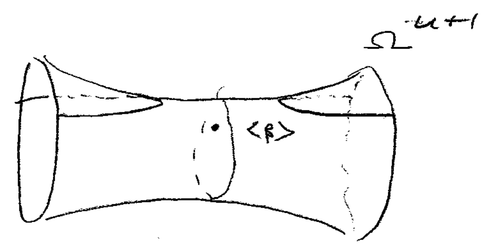
$$= -2(\text{const})^2 \underbrace{(e_0|e_s)}_{-\frac{1}{2}\lambda_0} \underbrace{(e_0|\alpha)}_{-\frac{\lambda_0}{2}\alpha^r} \underbrace{(e_s|\alpha)}_{-\frac{\lambda_0}{2}\alpha^s} a$$

$$(\beta|\beta)/\lambda_0 = \left(\text{const}\,\frac{\lambda_0}{2}\right)^2 \alpha^r \alpha^s a$$

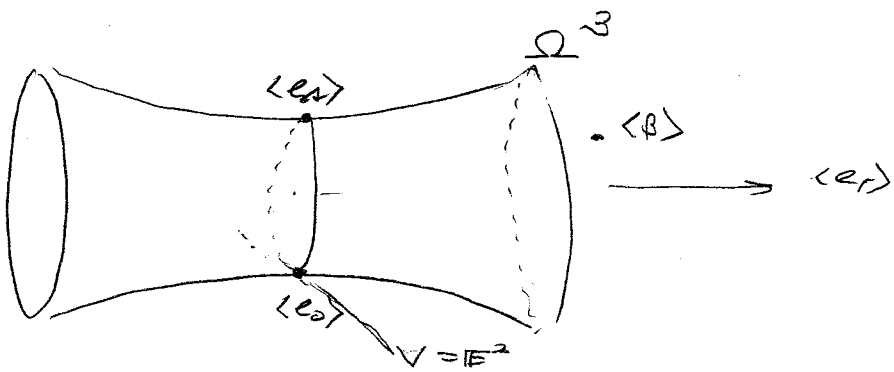
Search for meridial Geodesics

Need $\langle\beta\rangle \in \Omega^-$ i.e. $(\beta|\beta)/\lambda_0 < 0$.





$n=2$ Euclidean plane



$$\left.\begin{aligned} \beta &= e_0 + e_r R + e_\infty S \\ (\beta|\beta)/\lambda_0 &= -R^2 - S \end{aligned}\right\} \begin{aligned} &\langle\beta\rangle \in \Omega^- \text{ for} \\ &R^2+S>0 \end{aligned}$$

Typical cycle of Ω^3:

$$\alpha = e_0 + e_1 a + e_2 b + e_r r + e_\infty(a^2+b^2-r^2)$$

We have

$$(\beta|\alpha)/\lambda_0 = -Rr - \tfrac{1}{2}(a^2+b^2-r^2) - \tfrac{1}{2}S.$$

$\langle\alpha\rangle \in \langle\beta\rangle^\perp$ or $(\beta|\alpha)=0$ gives

$$a^2+b^2-r^2+2Rr+S=0$$

$$a^2+b^2+S = r^2-2Rr$$

$$a^2+b^2+S+R^2 = (r-R)^2$$

$$r = R \pm \sqrt{a^2+b^2+R^2+S}$$

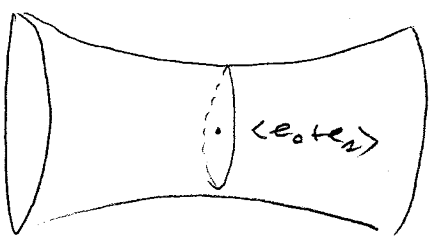
Cycle is $(x-a)^2 + (y-b)^2 = r^2$ ~~with~~ orientation:

$$x^2 - 2ax + y^2 - 2by = r^2 - a^2 - b^2$$

$$= (R \pm \sqrt{a^2+b^2+R^2+S})^2 - a^2 - b^2$$

$$= 2R^2 + S \pm 2R\sqrt{a^2+b^2+R^2+S}$$

$$\underset{\text{say }+}{r = R +} \sqrt{a^2+b^2+\underbrace{R^2+S}_{\substack{>0 \ \Omega^- \\ <0 \ \Omega^+}}} \quad \begin{matrix} > R + \sqrt{a^2+b^2} & \text{for } \Omega^- \\ < R + \sqrt{a^2+b^2} & \text{for } \Omega^+ \end{matrix}$$

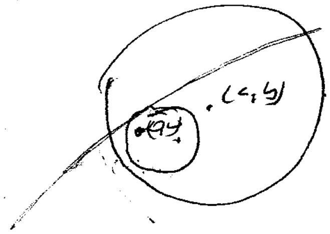


$(e_0 \mid e_2) = -\frac{1}{2}$ $\quad (\lambda_0 = 1)$

$(e_0 + e_2 \mid e_0 + e_2) = -1$

$R = 0,\ S = 1.$

$$r = 0 \pm \sqrt{a^2+b^2+0^2+1} = \pm\sqrt{a^2+b^2+1}$$



$$(x-a)^2 + (y-b)^2 = a^2 + b^2 + 1$$

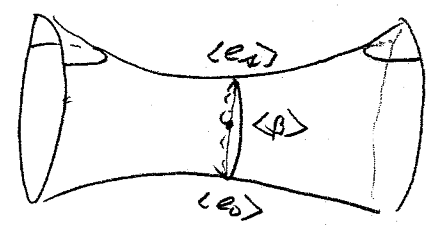
$$x^2 + y^2 - 2ax - 2by = 1$$

$\cdot\ (a,b)$

$\beta = e_0 + e_\infty S$

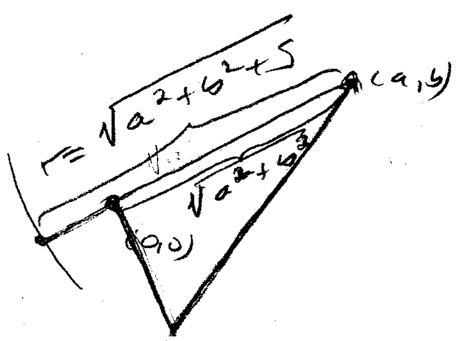
$(\beta|\beta)/\lambda_0 = -S$

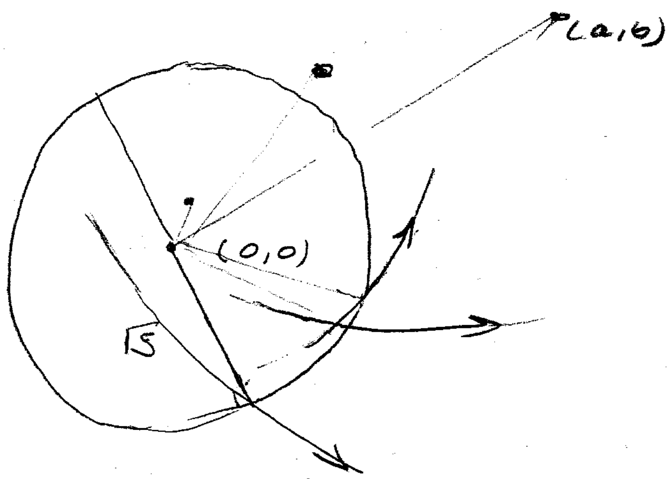
$r = \pm\sqrt{a^2+b^2+S}$



$\boxed{x^2+y^2-2ax-2by=S}$ with $S>0$ fixed.

$(x-a)^2+(y-b)^2 = a^2+b^2+S$





Fix a circle with center $(0,0)$ and radius $\sqrt{S}>0$. For every point (a,b), draw a cycle with center (a,b) passing through the diameter of length $\sqrt{S}$ perpendicular to line $(0,0)$ to (a,b). These cycles constitute an [illegible] bundle.