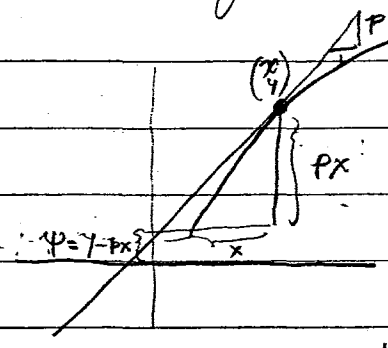
1-21-90

What I remember of Legendre Transform

from Thermo Based on Plücker duality



$y = f(x)$ $p = f'(x)$ $\psi = y - px$

$dy = p\,dx$ Let $x = h(p)$

$\psi = g(p) = f(h(p)) - p\,h(p)$

$d\psi = dy - p\,dx - x\,dp = -x\,dp$ $\frac{d\psi}{dp} = -x$

In thermo $L = L(x, \dot{x})$ $H = H(x,p) = L - p\dot{x}$ $p = \frac{\partial L}{\partial \dot{x}}$

$dL = \frac{\partial L}{\partial x}dx + \frac{\partial L}{\partial \dot{x}}d\dot{x} = \dot{p}\,dx + p\,d\dot{x}$

$\dot{p} = \frac{\partial L}{\partial x}$ $dH = dL - p\,d\dot{x} - \dot{x}\,dp = \dot{p}\,dx - \dot{x}\,dp$

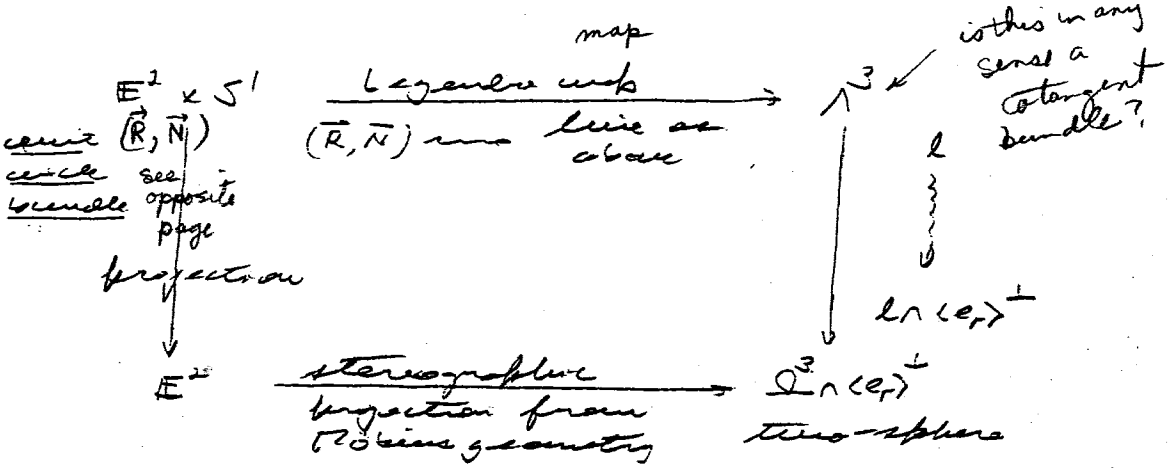
$dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial p}dp$ so $\frac{\partial H}{\partial x} = \dot{p}$ $\frac{\partial H}{\partial p} = -\dot{x}$

Let $\dot{x} = G(x,p)$

so $H(x,p) = L(x, G(x,p)) - p\,G(x,p)$

It takes the tangent bundle to the cotangent bundle

Aside One can also describe this as:

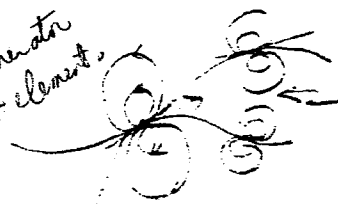


Note that on the left we use the Euclidean group $E(2,\mathbb{R}) = O(2,\mathbb{R}) \cdot \mathbb{R}^2$ of dimension 3, and on the right the Lie group $G = PO(4;\mathbb{R})$ of dimension 10. $\dim O(2;\mathbb{R}) = \frac{5 \cdot 4}{2} = 10$ ... so to $PO(2;\mathbb{R})_1$?

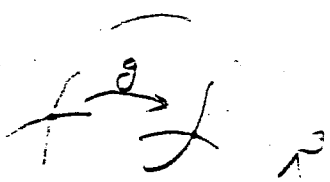
Key observation. Although Lie transformations are not point transformations of the Euclidean plane, curves can be sent to curves.
For: If $g \in G$, and $\vec{R}(t)$ a curve in $\mathbb{E}^2$, then g sends $\hat{R}(t)$ in Λ^3 to $g\hat{R}(t)$. The curve $g\hat{R}(t) \cap \langle e_1 \rangle^+$ is the image of $\vec{R}(t)$ under g.
see figure 1-17-90 on how Lie trans induce Möbius trans formations or harmonic homologies

Proposition. Tangent curves in $\mathbb{E}^2$ are sent to tangent curves by Lie transformations
Proof Curves are tangent when they share a contact element, i.e. the curves in Λ^3 must lie in a plane. This continues to hold under a Lie transformation

curves in Λ^3 are ruled surfaces. They are tangent when they share a generator as contact elements.



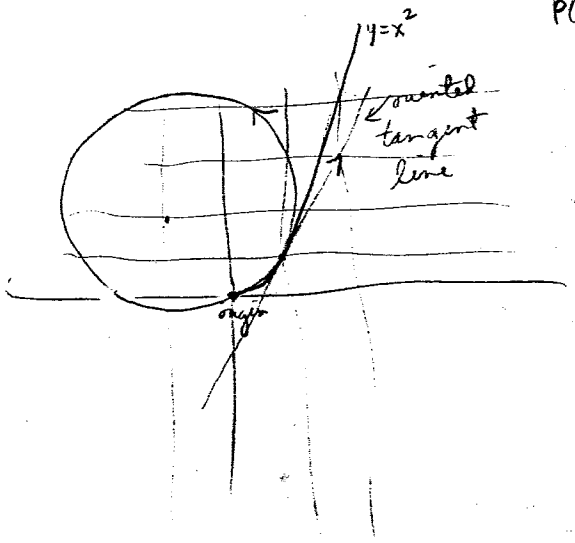
Curves can share a cycle, but not a contact element unless they are tangent



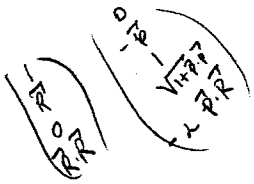
Cycles fixed by the Lie transformations
Contact elements are sent to contact elements

Why I think the sign is wrong Consider $\bar{x} = \begin{pmatrix} t \\ t^2 \end{pmatrix}$

$p(t) = 2t$ Let $t =$ $\bar{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $p(1) = 2$



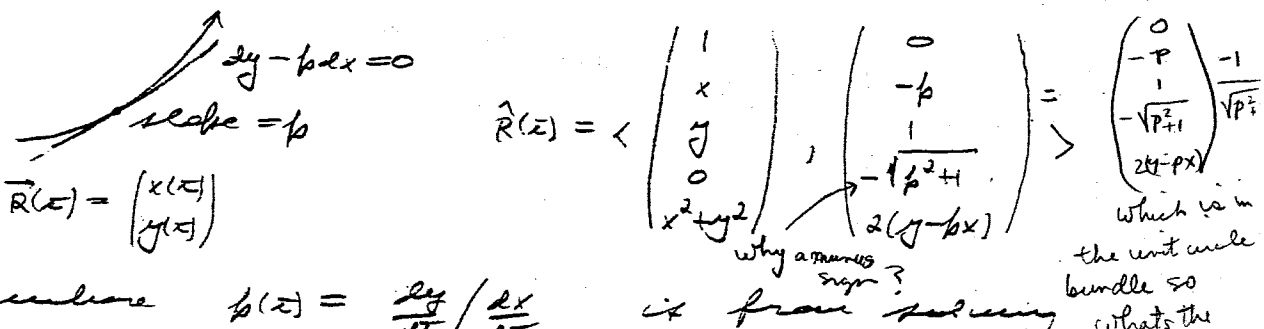
$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 1 \\ +\sqrt{5} \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ +\sqrt{5} \\ 0 \end{pmatrix}$$



For this reason, Lie transformations are called <u>contact transformations</u>.

<u>N.B.</u> Contact transformations do <u>not</u> operate on the points of E^2, <u>but</u> on the contact elements. and each contact element contains exactly one point.

<u>Remark</u> Lie used, not the unit circle bundle, but the "projective cotangent" bundle. And:



where $p(t) = \frac{dy}{dt} \Big/ \frac{dx}{dt}$ is from solving the <u>differential equation</u> $\frac{dy}{dx} = p(x, y)$

or integrating $dy - p\,dx$ on $\mathbb{R}^3$ with coordinates (x, y, p).

This was Lie's original motivation for using groups: to transform PDEs into simpler equations which can be integrated. For PDE's in $\mathbb{R}^4$, one chooses the <u>contact form</u>

$$dx_n - p_1\,dx_1 - \cdots - p_{n-1}\,dx^{n-1}$$

on the $(2n-1)$-dimensional manifold.

given $\underset{\sim}{L}$ st ${}^t\underset{\sim}{L}\underset{\sim}{G}\underset{\sim}{L} = \underset{\sim}{G}k^2$

$$\mathbb{E}^2 \times S^1 \longrightarrow \Lambda^3$$
$$(\vec{x}, \vec{y}) \longmapsto \ell = \{ \bar{X}\lambda^1 + \bar{Y}\lambda^2 \mid \bar{\lambda} \neq 0 \}$$

$$\bar{X} = \begin{pmatrix} 1 \\ \vec{x} \\ 0 \\ \vec{x}\cdot\vec{x} \end{pmatrix} \quad \bar{Y} = \begin{pmatrix} 0 \\ \vec{y} \\ 1 \\ 2\vec{x}\cdot\vec{y} \end{pmatrix} \qquad \bar{x}\cdot\bar{x} = \bar{Y}\cdot\bar{Y} = \bar{x}\cdot\bar{Y} = 0$$

$$\bar{\xi} = \underset{\sim}{L}\bar{X} = \begin{pmatrix} \xi^0 \\ \vec{\xi} \\ \xi^4 \end{pmatrix} \qquad \bar{\zeta} = \underset{\sim}{L}\bar{Y} = \begin{pmatrix} \zeta^0 \\ \vec{\zeta} \\ \zeta^4 \end{pmatrix}$$

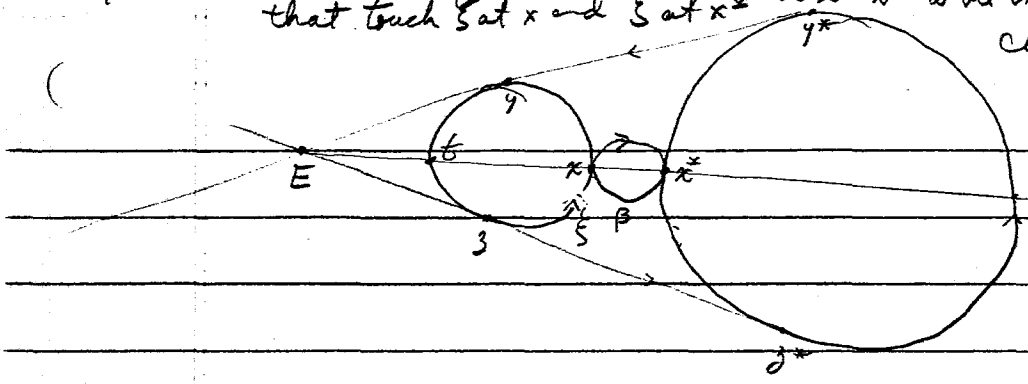
$$\bar{Z} = \bar{\xi}\zeta^4 - \bar{\zeta}\xi^4 \qquad \bar{Z}\cdot\bar{Z} = 0$$

$$\mathbb{E}^2 \times S^1 \longrightarrow \mathbb{E}^2$$
$$(\vec{x}, \vec{y}) \longmapsto \left(\frac{\vec{\xi}\zeta^4 - \vec{\zeta}\xi^4}{\xi^0\zeta^4 - \zeta^0\xi^4} \right)$$

image curve under a Lie trans. of a curve denoted by a curve in the unit tangent bundle.

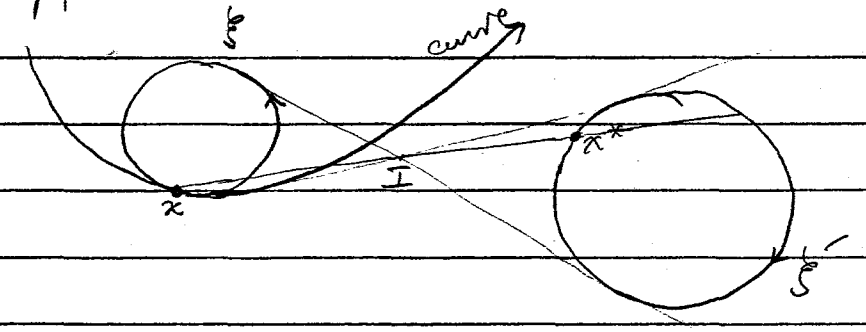
1-21-90

1) Lemma: If ξ is sent to ξ' by a Lie inversion and β is a fixed cycle that touch ξ at x and ξ' at x^* then x^* is the image of x wrt the Lie contact transformation induced by the Lie inversion.



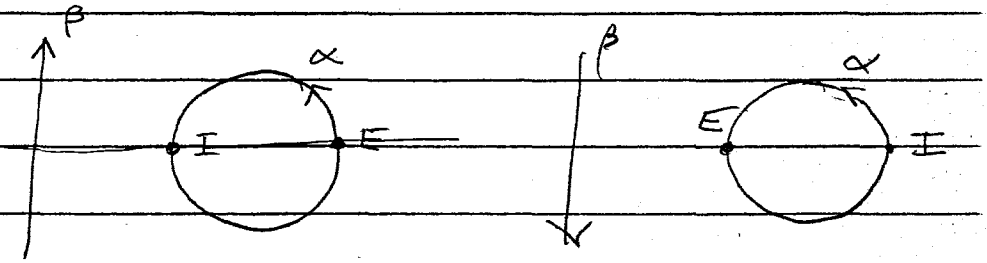
It restricts to oriented projection

Application:



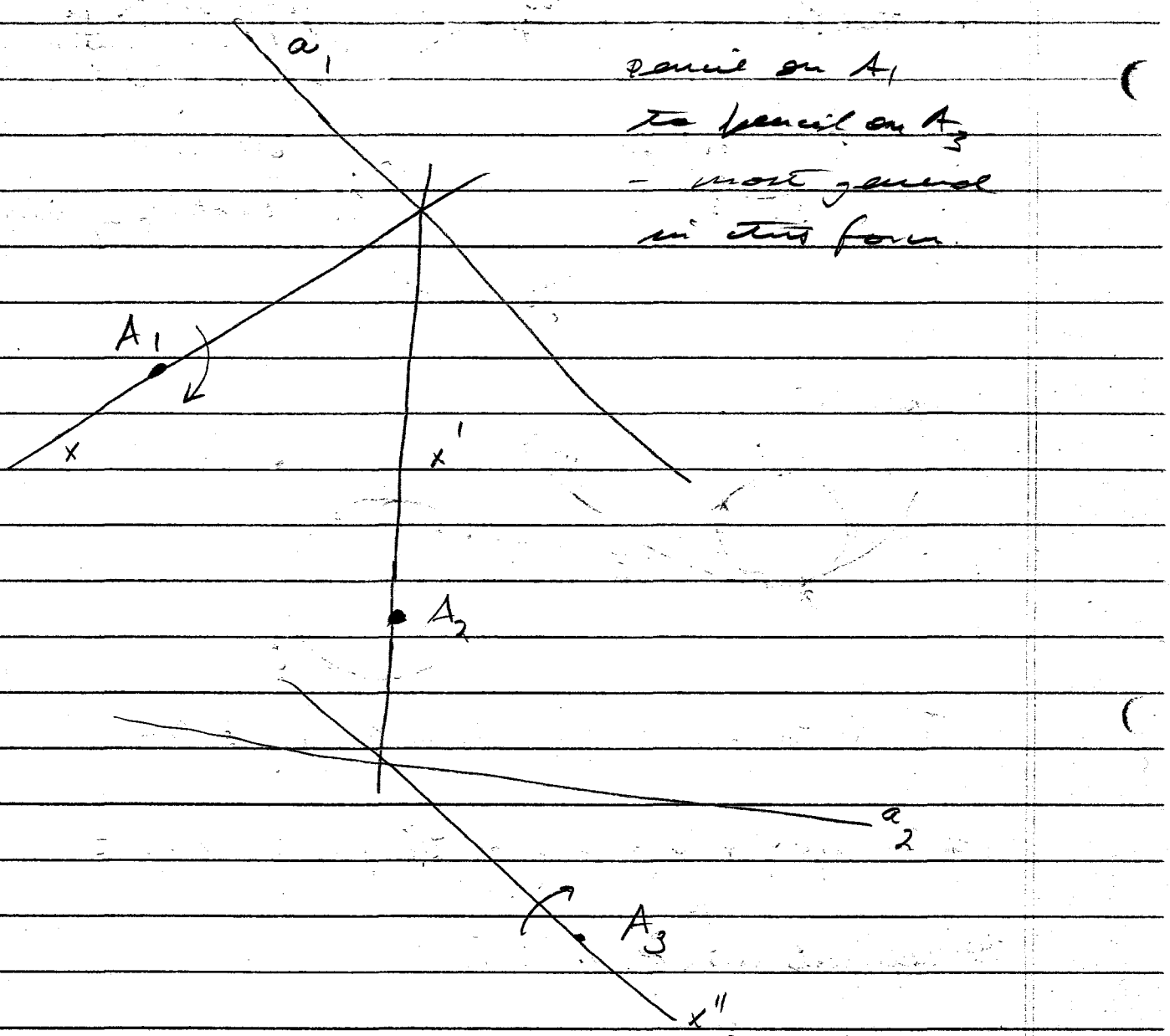
To calc the $*$-image of x calc the image of ξ then the $*$-image of x is given by oriented projection from I the appropriate center of similitude.

Note:



proj. beam:

pencil on A_1
to pencil on A_3
- most general
in this form



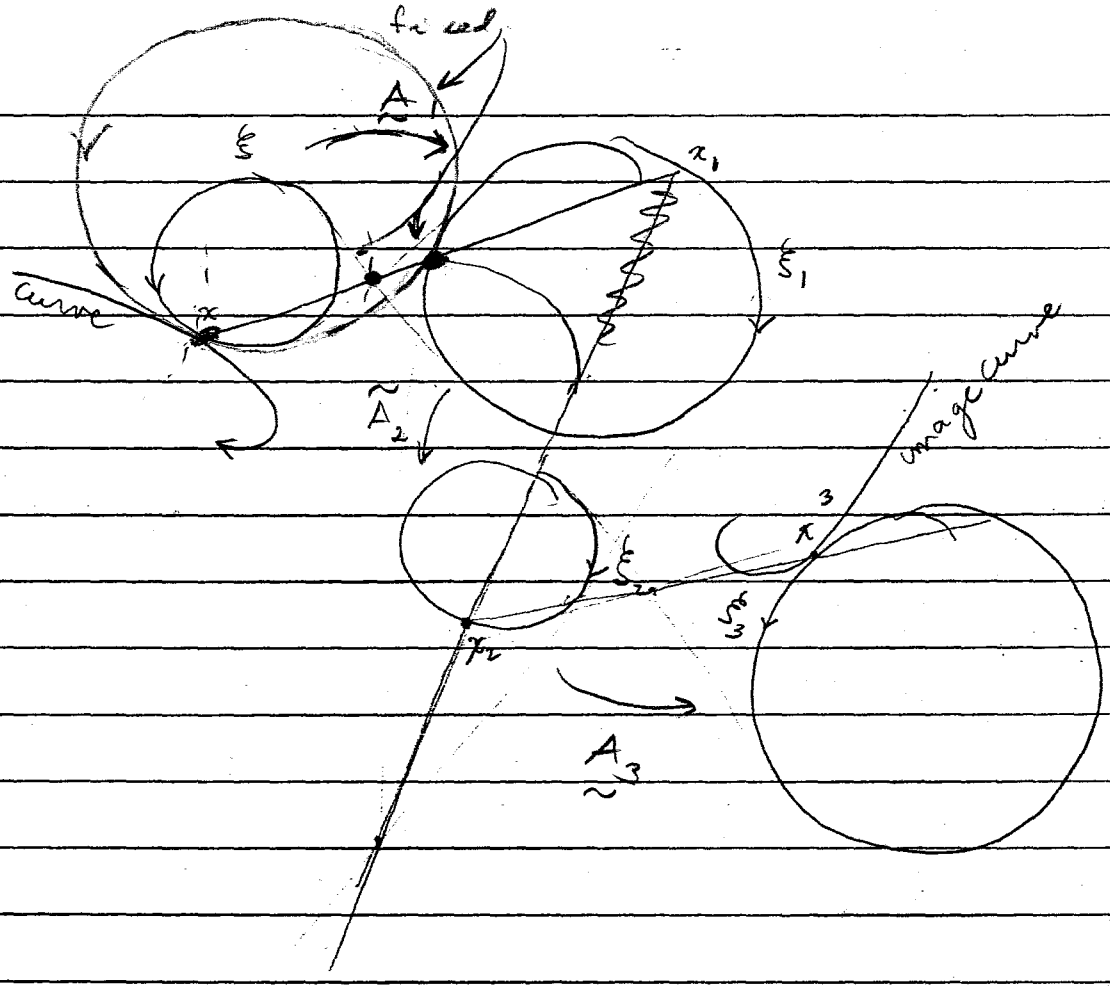
$(A_1, a_1) \leftrightarrow \underset{\sim}{A_1}$ on next page

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2)

$\underset{\sim}{L} = \underset{\sim}{A}_1 \underset{\sim}{A}_2 \underset{\sim}{A}_3$

$\underset{\sim}{A} = \underset{\sim}{id} - 2\dfrac{\bar{A}\,\bar{\bar{A}}\,\underset{\sim}{G}}{\bar{\bar{A}}\,\underset{\sim}{G}\,\bar{A}}$



The is the Lie contact trans analogue of describing the general collineation as a product of polarities.

Use centers of similitude to find point in a contact element

See pictures re α_1 & α_2 1/17/90

Lie int $\Sigma \to \Sigma'$ induces a homology on pts of Σ to pts of Σ' which is a "oriented projection".

1-24-90

$C = \cosh t \quad S = \sinh t$

example

$$\underset{\sim}{\alpha} = \left(\begin{array}{c|ccc|c} 1+C & 0 & 0 & -2S & 1-C \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ -S & 0 & 0 & 2C & S \\ \hline 1-C & 0 & 0 & 2S & 1+C \end{array}\right)$$

contact element $(\bar{\pi}, \bar{\gamma}) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \\ 0 & 1 \\ 5 & 1 \end{pmatrix}$

$$\underset{\sim}{\alpha}(\bar{\pi}\,\bar{\gamma}) = (\bar{\pi}'\bar{\gamma}') = \left(\begin{array}{cc} 2(3-2C) & 2(1-S) \\ 2 & 2 \\ 4 & 2 \\ 4A & 2C \\ 4C & 2(1+S) \end{array}\right) \begin{pmatrix} C-2S \\ 2C-2S \end{pmatrix} \frac{1}{3C-2C^2-2S+2S^2}$$

point

image of $\bar{\pi}$ $\quad \bar{\pi}'(t) = \begin{pmatrix} e^{-t}-\sinh t \\ 2e^{-t} \end{pmatrix} \frac{1}{\cosh t - 2e^{-t} - 2}$

Example of a W-curve.