ca. 1977

On LIE's Higher Sphere Geometry
– Colloquium talk.

1. Introduction.

Old subject (LIE, 1872) from modern viewpoint. Many questions remain to be understood in modern terms. Spaces & transformations – from old books on "higher geometry".

1) Sphere $x^2+y^2+z^2-2ax-2by-2cz+C=0$ in cplx or real E^3. Coordinates: a, b, c, C, r with $r^2=a^2+b^2+c^2-C$. Sign of r ⇔ "orientation" of sphere. Homog.: $a=\frac{\alpha}{\nu}$, $b=\frac{\beta}{\nu}$, $c=\frac{\gamma}{\nu}$, $r=\frac{\lambda}{\nu}$, $C=\frac{\mu}{\nu}$.
Ψ^4: $\alpha^2+\beta^2+\gamma^2-\lambda^2-\mu\nu=0$ in P^5.
LIE's quadric: Ψ^4 = space of oriented spheres (& planes, $\nu=0$) in E^3.
Tangency of oriented spheres:
$$(a-a')^2+(b-b')^2+(c-c')^2=(r-r')^2$$
$$2aa'+2bb'+2cc'-2rr'-C-C'=0$$
$$2\alpha\alpha'+2\beta\beta'+2\gamma\gamma'-2\lambda\lambda'-\mu\nu'-\nu\mu'=0.$$
Spheres in E^3 are tangent iff corresp. pts. on Ψ^4 are conjugate.

2) Line $(x_0,\dots,x_3)$ to $(y_0,\dots,y_3)$ in real or cplx P^3. PLÜCKER coords.:
$p_{ij}=x_iy_j-y_ix_j$ $\quad \xi_1=p_{12}$, $\xi_2=p_{31}$, $\xi_3=p_{23}$,
– homogeneous. $\quad \xi_4=p_{03}$, $\xi_5=p_{02}$, $\xi_6=p_{01}$.
Ω^4: $\xi_1\xi_4+\xi_2\xi_5+\xi_3\xi_6=0$ in P^5.
PLÜCKER's quadric: Ω^4 = space of lines in P^3.
Intersection of lines:
$$\xi_1\xi_4'+\xi_2\xi_5'+\xi_3\xi_6'+\dots+\xi_6\xi_3'=0.$$
Lines in P^3 intersect iff corresp. pts on Ω^4 are conjugate.

3) Line-sphere transformation

$\xi_1=\alpha+\sqrt{-1}\beta \quad \xi_4=\alpha-\sqrt{-1}\beta$ Discovered by LIE.
$\xi_2=\gamma+\lambda \quad \xi_5=\gamma-\lambda$ KLEIN's form.
$\xi_3=\mu \quad \xi_6=-\nu$ Takes Ψ^4 to Ω^4.

Conjugate pts. sent to conjugate pts, or: Tangent oriented spheres of E^3 sent to intersecting lines of P^3. Contact trf.

Questions and remarks.

1) Old books: Contact trf on 5-dim'l (x,y,z,p,q) preserves $\omega=dz-p\,dx-q\,dy$ up to non-vanishing scalar multiple. Coords. (x,y,z,p,q) describe surface elt. $z'-z=p(x'-x)+q(y'-y)$ at (x,y,z) in E^3. $\omega=0$: At infinitesimally adjacent pts., pt. of one element lies on plane of other. Line-sphere trf: What is 5-dim'l space? What is ω? Is ω preserved?

2) Line-sphere trf contains $\sqrt{-1}$: $\xi_1=\alpha+\sqrt{-1}\beta$, etc. No problem for cplx E^3 & P^3. What about real E^3 & P^3?

3) KLEIN, Evanston Colloquium, 1893. "... by means of [the line-sphere trf.] the lines of curvature of a surface are transformed into asymptotic lines of the transformed surface, and vice versa. ... This must certainly be regarded as one of the most elegant contributions to differential geometry in recent times." – p. 17.

4) KLEIN, Ibid. Regarding the line-sphere trf & contact trfs: "It has been the final aim of LIE from the beginning to make progress in the theory of [partial] differential equations." – p. 24.

Answers to 1) & 2) follow, but not 3) & 4).
N.B. Answers to 1) & 2) are exactly as one would expect – even from 19th century viewpoint.
And! Answers are in terms of homogeneous contact manifolds.
Answers question of S. SASAKI, 1965.

Place on blackboard in advance.

2. Homogeneous contact manifolds.

Theory of BOOTHBY (1961) & WOLF (1965)

→ Contact manifold: M^{2n-1}, $\{(U_\alpha, \omega_\alpha)\}$

i) $\omega_\alpha \wedge (d\omega_\alpha)^{n-1} \neq 0$ on U_α

ii) $\omega_\alpha = f_{\alpha\beta}\,\omega_\beta$ on $U_\alpha \cap U_\beta$

iii) $\{(U_\alpha, \omega_\alpha)\}$ maximal

Complex. Note: This def'n is very classical.

Contact trf: $g : M \to M'$: $\{(g^{-1}U'_\alpha, g^*\omega'_\alpha)\} \subset \{(U_\alpha, \omega_\alpha)\}$

Bundle formulation

$B^{2n} \xrightarrow{\mathbb{C}^*} M^{2n-1}$, ω on B

a) $(d\omega)^n \neq 0$ on B

b) $\omega = 0$ on fibers

c) $R_a^*\omega = a\omega$, $a \in \mathbb{C}^*$.

ω_α's are ω pulled down by sections.

Contact trf: $g : B \to B'$, $g^*\omega' = \omega$

Classical example V^n

B = cotangent bdle of V^n

M^{2n-1} = projective cotgt bdle

$\xi = \sum_{i=1}^n u_i(\xi)\,dx_i$, $\xi \in B$

$\omega = \sum_1^n u_i\,dx_i$.

Homogeneous contact manifolds

G linear algebraic group, effective & transitive on compact algebraic M.

i) G is transitive on B.

ii) G is semi-simple

iii) G has trivial center.

M & G connected. So G is of adjoint type.

So: $M = G/P$, P parabolic, $B = G/P_1$,

$P_1 = \ker \chi$, $g b_0 = R_a b_0$, $a = \chi(g)$, $g \in P$.

Lie algebra level ω_0 form on $\underline{g}$

ω (pulled back to G) $= \omega_0(g^{-1}dg)$.

a) $(d\omega_0)^n \neq 0$. $d\omega_0(X,Y) = -\frac{1}{2}\omega_0([X,Y])$.

b) $\omega_0 = 0$ on $\underline{p}$

c) $\omega_0(g^{-1}Xg) = \chi(g)\,\omega_0(X)$, $g \in P$, $X \in \underline{g}$

or ${}^*\mathrm{Ad}(g)\,\omega_0 = \chi(g)\,\omega_0$, $g \in P$.

Equivalent. $\omega_0(X) = \langle W, X\rangle$ (KILLING)

a) $Z_{\underline{g}}(W) = \underline{p}_1$, centralizer.

b) $W \perp \underline{p}$

c) $\mathrm{Ad}(g)W = \chi(g)W$, $g \in P$

or $[X, W] = \chi'(X)\,W$, $X \in \underline{p}$

So: $\rho = \chi'$ is root of $\underline{g}$ wrt $\underline{h} \subset \underline{p}$; $E_\rho = W$ root vector.

Description. Using classification of parabolic subalgebras, one shows: ρ is maximal root, so $\underline{g}$ is simple.

$\underline{p} = \underline{h} + \sum \underline{g}_\alpha$, sum over $\langle H_\rho, H_\alpha\rangle \geqslant 0$.

$\underline{p}_1$: $X \in \underline{p}$ and $\langle H_\rho, X\rangle = 0$.

$\omega_0(X) = \langle E_\rho, X\rangle$

G = connected centerless, LIE alg. $\underline{g}$.

$M = G/P$, $B = G/P_1$

G = identity component of group of contact automorphisms of M.

Classification. Above steps reversible, so alg. homog. cplx contact manifolds correspond to simple cplx LIE algebra types: $A_n, \dots, G_2$.

Example: A_3. Extension to A_n, $n \geqslant 2$, is evident.

Formal part. Take

$G = PSL(4;\mathbb{C}) = SL(4;\mathbb{C})/\{\pm 1_4, \pm i 1_4\}$

$\underline{g}$ = 4×4 matrices of trace 0

$\underline{h}$ = diagonal matrices of trace 0

$\delta_i(H)$ = i^{th} entry of $H \in \underline{h}$, lin. form.

$\delta_0 + \delta_1 + \delta_2 + \delta_3 = 0$.

Roots: $\delta_i - \delta_j$, $i \neq j$, $E_{\delta_i - \delta_j} = E_{ij}$ i in i row, j col.

Simple roots: $\delta_0 - \delta_1, \dots, \delta_2 - \delta_3$

Max'l root: ρ = sum of simple $= \delta_0 - \delta_3$

Killing: $\langle X, Y\rangle = 8\,\mathrm{tr}(XY)$, use $\mathrm{tr}(XY)$.

$H_\rho = \mathrm{diag}(1, 0, 0, -1)$, etc., $E_\rho = E_{03}$

$\langle H_\rho, H_{\delta_i - \delta_j}\rangle < 0$ for $j = 0$ or $i = 3$ only

$\underline{p}$ consists of matrices $\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$. $\underline{p}_1$ is not explicitly needed.

$\omega_0(X) = \langle E_{03}, X\rangle$ = 30-entry of $X \in \underline{g}$

To identify the spaces: x = col. vector = pt. of P^3, u = row vector = plane of P^3.

(x, u) = incident pt.-plane pair = surface element of P^3. $ux = 0$.

G is transitive on incident (x,u)

$g\cdot(x,u) = (gx, ug^{-1})$, $ux=0$.

P = isotropy subgroup of (x_0,u_0)

$x_0 = {}^t(1,0,0,0)$, $u_0 = (0,0,0,1)$. x_0 lies on plane $u_0x'=0$ or $x'_3=0$. or $Z'=0$.

P has $\mathfrak{p}$ above as its LIE algebra.

G/P = incident pt.-plane pairs of P^3

  = projective cotgt bdle of P^3, dim $=5$

G/P_1 = cotangent bdle of P^3

<u>Classical identification.</u>

Set
$$X = \begin{pmatrix} 0 & & & \\ x & & \bigcirc & \\ y & & & \\ z-\frac{1}{2}(px+qy) & p & q & 0 \end{pmatrix} \text{ nilpotent matrix, of } \mathfrak{g}.$$
$g = \exp X$.

$(x,y,z,p,q) \to g \bmod P$ gives chart $\mathbb{C}^5 \to U$ (open) $\subset G/P$, coordinates.

$g\cdot(x_0,u_0) = (x,u)$, $g = \exp X$ as above, gives

$x = {}^t(1,x,y,z)$, $u = (-z+px+qy, -p, -q, 1)$.

$ux'=0$ is $z'-z = p(x'-x)+q(y'-y)$,

$x' = {}^t(1,x',y',z')$, so (x,u) corresponds to (x,y,z,p,q) in classical sense.

$$\begin{tikzcd} & B = G/P_1,\ \omega \arrow[d, "\mathbb{C}^*"] \\ \mathbb{C}^5 \arrow[ur, "(x,y,z,p,q)\to g \bmod P_1"] \arrow[r, "(x,y,z,p,q)\to g \bmod P"'] & U \subset G/P \arrow[u, bend left, "\text{section}"] \end{tikzcd}$$

$g = \exp X$ as above. $g \bmod P \to g \bmod P_1$ is section of B over U. ω pulled down to U in coords. (x,y,z,p,q) is ω pulled back to $\mathbb{C}^5$ by upper map: $\omega = \omega_0(g^{-1}dg)$, $g = \exp X$, X as above.

$$g^{-1}dg = \frac{1-e^{-\operatorname{ad} X}}{\operatorname{ad} X}(dX) = \begin{pmatrix} 0 & & & \\ dx & & \bigcirc & \\ dy & & & \\ dz-p\,dx-q\,dy & dp & dq & 0 \end{pmatrix}$$

$\omega = \omega_0(g^{-1}dg)$ = 30-entry of $g^{-1}dg$

 $= dz - p\,dx - q\,dy$, classical.

<u>N.B.</u> This establishes what is meant by classical identification of coordinates and contact form.

3. <u>Sphere geometry.</u>

<u>Line geometry</u> - the key.

$A_3 \simeq D_3$, simple cplx LIE algebras, so can express space of surface elements of P^3 in terms of D_3.

<u>Classical correspondence</u>: P^3 and Ω^4

line	point	Double ruling
point (point star)	plane (1^{st} family)	
plane (ruled plane)	plane (2^{nd} family)	
surface element (incident pt & plane)	line (intersection of plane of each family)	

So:
$$\begin{pmatrix}\text{space of surface}\\ \text{elements of } P^3\end{pmatrix} \simeq \begin{pmatrix}\text{space of}\\ \text{lines in } \Omega^4\end{pmatrix}, \text{ dim} = 5.$$

This leads to explicit isomorphism $A_3 \simeq D_3$ or $\mathfrak{sl}(4;\mathbb{C}) \simeq \mathfrak{o}(3,3;\mathbb{C})$.

<u>Description</u> of A_3 by D_3. Apply explicit formulas for isom. to obtain:

$\mathfrak{g} = \mathfrak{o}(A;\mathbb{C})$ $A = \begin{pmatrix} 0 & 1_3 \\ 1_3 & 0 \end{pmatrix}$. Use same letters as before.

$G = PSO(A;\mathbb{C})$

(x_0,u_0) above is represented on Ω^4 by line l_0 joining ${}^t(000010)$ and ${}^t(000001)$.

$\mathfrak{p}$ = isotropy subalgebra of l_0

P = isotropy subgroup of l_0.

G is transitive on lines of Ω^4

G/P = space of lines of Ω^4, dim $=5$.

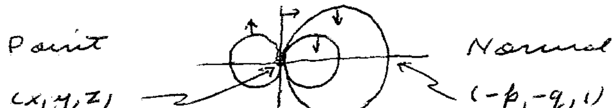
Explicit isomorphism also makes correspond: ρ, $E_\rho = W$, ω_0, ω, so:

$$\begin{pmatrix}\text{space of}\\ \text{surface}\\ \text{elts. of } P^3\end{pmatrix} \overset{\text{isom}}{\simeq} \begin{pmatrix}\text{space}\\ \text{of lines}\\ \text{in } \Omega^4\end{pmatrix}$$ Isom. of homog. alg. contact manifolds.

<u>Note</u>. Hereafter G, P, etc. unprimed refer to $G = PSO(A;\mathbb{C})$, etc. for Ω^4.

<u>Sphere geometry</u> Key: lines of Ψ^4.

<u>Geometric part.</u> A line of Ψ^4 corresponds to a pencil of mutually tangent oriented spheres (and planes) of E^3 - an oriented surface element. Include these "at infinity".



The pencil

$(x'-x+\tau p)^2+(y'-y+\tau q)^2+(z'-z-\tau)^2=\tau^2(p^2+q^2+1)$

contains the plane ($\tau=\infty$)

$$z'-z=p(x'-x)+q(y'-y)$$

Esp. the line ℓ_0' of Ψ^4 joining $^\tau(000001)$ and $^\tau(001-100)$ corresponds to the pencil

$$x'^2+y'^2+(z'-\tau)^2=\tau^2$$

containing the plane $z'=0$ through the origin $x=0, y=0, z=0$.

$G' = PSO(A';\mathbb{C})$ $\quad A'=\left(\begin{array}{c|ccc} 2\cdot 1_3 & & 0 & \\ \hline & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ & 0 & -1 & 0 \end{array}\right)$

$\underline{g}' = \underline{o}(A';\mathbb{C})$

P' = isotropy subgroup of ℓ_0'

$\underline{p}'$ = isotropy subalgebra of ℓ_0'

G' is transitive on lines of Ψ^4

G'/P' = space of lines of Ψ^4, dim = 5

= space of pencils of E^3

= space of oriented surface elts. of E^3

<u>Classical identification.</u> The following may be obtained by very complicated direct calculation: $\underline{h}'$, P' so $\underline{p}'$ is $\underline{p}'$ above, $\underline{p}_1'$, $E_{\rho}'=W'$ giving contact form. $X'(x,y,z,p,q)$ nilpotent matrix of $\underline{g}'$, entries essentially quadratic in $x,\dots,q$ so that $g'\ell_0'$, $g'=\exp X'$, is the line of Ψ^4 corresponding to the pencil above. Then $(x,y,z,p,q)\to g' \bmod P'$ gives chart $\mathbb{C}^5 \to U'$ (open) $\subset G'/P'$, classical coords,

$$\begin{tikzcd} & B'=G'/P_1',\ \omega' \arrow[d, "\mathbb{C}^*"] \\ \mathbb{C}^5 \arrow[ur, "(x,y,z,p,q)\to g' \bmod P_1'"] \arrow[r, "(x,y,z,p,q)\to g' \bmod P'"'] & U' \subset G'/P' \arrow[u, bend right, "\text{section}"'] \end{tikzcd}$$

ω' pulled down to U' by section of B' over U' is

$$\omega'=\omega_0'(g'^{-1}dg')=\frac{1-e^{-\operatorname{ad}X'}}{\operatorname{ad}X'}(dX')$$
$$= dz - p\,dx - q\,dy \quad (\text{up to scalar} \neq 0)$$

This establishes classical identification of coords. & contact form.

<u>Note</u> Extension to sphere geometry of E^n, i.e. space of lines of Ψ^{n+1};

$$\alpha_1^2+\alpha_2^2+\cdots+\alpha_n^2-\lambda^2-\mu\nu=0$$

requires considering B_ℓ, $n+3=2\ell+1$, n even, and D_ℓ, $n+3=2\ell$, n odd. Geometric results are uniform.

<u>Line-sphere transformation.</u>

Classically a change of quad. form

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{pmatrix} = T \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \lambda \\ \mu \\ \nu \end{pmatrix}, \quad T=\left(\begin{array}{cc|cccc} 1 & \sqrt{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & -\sqrt{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

$A' = {}^\tau TAT$, so $G'=T^{-1}GT$ & $\underline{g}'=T^{-1}\underline{g}T$.

$\Psi^4 = T^{-1}\Omega^4$ in P^3, (lines of Ψ^4) $= T^{-1}$(lines of Ω^4).

$\ell_0'=T^{-1}\ell_0$, so $P'=T^{-1}PT$ & $\underline{p}'=T^{-1}\underline{p}T$.

All parts of algebraic construction similarly correspond; esp. $W'=T^{-1}WT$. ω on $B=G/P_1$ corresponds to ω' on $B'=G'/P_1'$.

$$\begin{tikzcd} (\text{lines of } \Omega^4) \arrow[r, "T^{-1}"] \arrow[d, equal] & (\text{lines of } \Psi^4) \arrow[d, equal] \\ G/P \arrow[r] & T^{-1}GT/T^{-1}PT = G'/P' \end{tikzcd}$$

$g \bmod P \to T^{-1}gT \bmod P'$ is a contact trf, modern sense.

4. <u>Real forms.</u>

<u>Example</u>: A_3 - the key.

View all quantities as real.

$G=PSL(4;\mathbb{C})$ is replaced by $G_0=PSL(4;\mathbb{R})$, real form wrt. matrix conjugation $g\to\bar{g}$. Max'l root ρ is real.

G_0/P_0 = incident point plane pairs of P_0^3

= proj. cotgt bdle of real P_0^3, dim/$\mathbb{R}$ = 5

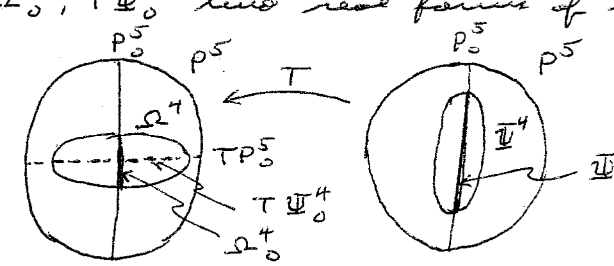
$P_0 = G_0 \cap P$. ω is real.

<u>General.</u> From WOLF (1969) can deduce:
G_0 real form of G wrt. $g \to cg$
Equivalent conditions:
(i) algebraic G/P is defined over $\mathbb{R}$ and G_0/P_0 = real points, $P_0 = G_0 \cap P$
(ii) $cP = P$
(iii) ρ, defining P, is real root, ${}^c\rho = \rho$.
${}^c\rho(H) = \overline{\rho(cH)}$, $H \in \mathfrak{h}$, $c\mathfrak{h} = \mathfrak{h}$.
G_0/P_0 is fixed points of conjugation $g \bmod P \to cg \bmod P$ on G/P.
From ρ real wrt. c, obtain real form of contact manifold:
$B_0 \to G_0/P_0$ real, ω real on B_0.

<u>Real line geometry.</u>
View previous quantities as real.
$G_0 = PSO(A;\mathbb{R})$, matrix conjugation
$\Omega^4_0 = P^5_0 \cap \Omega^4$, real PLÜKER quadric.
G_0/P_0 = lines of Ω^4_0, $\dim/\mathbb{R} = 5$

<u>Real sphere geometry</u>
View previous quantities as real.
$G_0' = PSO(A';\mathbb{R})$, matrix conjugation
$\Psi^4_0 = P^5_0 \cap \Psi^4$, real LIE quadric.
G_0'/P_0' = lines of Ψ^4_0, $\dim/\mathbb{R} = 5$.

<u>Real forms of quadrics.</u>
T = line-sphere transformation
Ω^4_0, $T\Psi^4_0$ two real forms of Ω^4.



<u>Real forms of contact manifolds.</u>
$G' = T^{-1}GT$ as before, $G = TG'T^{-1}$.
$TG_0'T^{-1}$ = real form of G for matrix conjugation on G' transported to G by T:

$$cg = T(\overline{T^{-1}gT})T^{-1} = S^{-1}\bar{g}S. \qquad S = \bar{T}T^{-1} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1_2 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_2 \end{array}\right).$$

$$\left.\begin{array}{ll} G_0, & g \to \bar{g} \\ TG_0'T^{-1} \simeq G_0', & g \to cg \end{array}\right\} \text{two distinct real forms of } G.$$

$$\left.\begin{array}{l} G_0/P_0 \\ TG_0'T^{-1}/TP_0'T^{-1} \simeq G_0'/P_0' \end{array}\right\} \text{two distinct real forms of } G/P.$$

The maximal root ρ for G/P is real for both $g \to \bar{g}$ and $g \to cg$.
<u>Conclusion:</u> Real line geometry and real sphere geometry are two distinct real forms of complex line geometry, as algebraic homogeneous contact manifolds. The classical descriptions place one of these two real forms in emphasis. The line-sphere transformation connects these two descriptions.
<u>Note</u> concurrence with classical view.

5. <u>Conclusion.</u>
Geometric descriptions of the complex alg. homog. contact manifds.

A_n: $\begin{pmatrix}\text{incident pt.-}\\ \text{hyperplane}\\ \text{pairs in } P^n\end{pmatrix} = \begin{pmatrix}\text{projective}\\ \text{cotangent}\\ \text{bdle of } P^n\end{pmatrix}$

B_ℓ, D_ℓ: $\begin{pmatrix}\text{space of lines}\\ \text{in a quadric}\end{pmatrix} = \begin{pmatrix}\text{LIE's higher}\\ \text{sphere geom.}\end{pmatrix}$

C_ℓ: complex projective space $P^{2\ell+1}$

E_6, E_7, E_8, F_4, G_2: unknown.