Math 151 — 21 May 1971

Spring 1971

Problems to be turned in: Due 4 June 1971

(A.9) a) Let $C(s)$ be a geodesic on an abstract surface and let $C_\varepsilon(s)$ be a differentiable family of geodesics with $C_0 = C$ (a variation of C through geodesics); s denotes arc length on each curve C_ε. Assume that the variation vector field $V(s) = \left(\frac{\partial C_\varepsilon(s)}{\partial \varepsilon}\right)_{\varepsilon=0}$ is normal to C along C. (Such variations can be constructed.) Let N be the normal field along C so that $V(s) = \phi(s) N(s)$. Show that $\phi(s)$ satisfies Jacobi's equation: $\frac{d^2\phi}{ds^2} + K(s)\phi = 0$ along C, where $K(s)$ denotes Gauss curvature at the point $C(s)$. (Hint: If $C_\varepsilon(s)$ is given by $u^i = u^i(s,\varepsilon)$ in local coordinates, differentiate the equation for geodesics

$$\frac{\partial^2 u^i(s,\varepsilon)}{\partial s^2} + \sum_{j,h=1}^{2} \Gamma^i_{jh}\left(u^1(s,\varepsilon), u^2(s,\varepsilon)\right) \frac{\partial u^j(s,\varepsilon)}{\partial s} \frac{\partial u^k(s,\varepsilon)}{\partial s} = 0$$

with respect to ε. Express the result in terms of geodesic parallel coordinates (Text, Ch. VII, Sect. 11, p. 174 ff) in which the metric is $du^2 + G(u,v)\,dv^2$, C is described by $u = s$, $v = 0$, and $V = \frac{\phi}{\sqrt{G}} \frac{\partial}{\partial v}$ along C.)

"TEXT" REFERS TO: J. J. STOKER, DIFFERENTIAL GEOMETRY, WILEY-INTERSCIENCE, NEW YORK, 1969.

(⑧ continued)

6) Show that Jacobi's equation has the qualitative interpretation: Nearby geodesics "converge" or "diverge" according as $K > 0$ or $K < 0$. (Compare Jacobi's equation with the equation $\frac{d^2 f}{ds^2} + K_0 f = 0$, $K_0 = \text{constant}$, whose solutions can be written down explicitly. Cf: The proof of the Sturm-Liouville Comparison Theorem, Text, p. 228-229.)

Remark. Jacobi's equation is important in the study of geodesics in the large. See Stoker, Ch. VIII, Sect. 12, pp. 247-254.

(A.10) (11) a) Let S be a closed regular surface in space, K Gauss curvature on S. Let $\{K \geq 0\}$ denote the set of points of S where $K \geq 0$. Show that $\int_{\{K \geq 0\}} K\, dA \geq 4\pi$, where dA is the area element of S. (Hint: $K\,dA$ is the area element for the spherical image of S; it suffices to show that every point of the unit sphere appears at least once as the unit normal to S at a point where $K \geq 0$. Bring an appropriate plane up from infinity.)

(⑨ continued)

b) Let $X = X_C(s)$ be a closed regular curve C in space; s arc length along C, L the length of C, and $\kappa(s)$, $N_C(s)$, $B_C(s)$ the curvature, principal normal, binormal of C respectively. Construct a tube surface S around C by $X = X(s,\theta) = X_C(s) + \varepsilon(N_C(s)\cos\theta + B_C(s)\sin\theta)$. Show: For $\varepsilon > 0$ small, S is a closed regular surface; and that the curvature and area element of S are given by

$$K = -\frac{\kappa(s)\cos\theta}{\varepsilon(1-\varepsilon\kappa(s)\cos\theta)}, \quad dA = \varepsilon(1-\varepsilon\kappa(s)\cos\theta)\,ds\,d\theta$$

respectively. Show that for this surface S:

$$\int_{\{K\geq 0\}} K\,dA = 2\int_0^L \kappa(s)\,ds.$$

c) Combine a) and b) to prove (after K. Voss) <u>Fenchel's Theorem</u>: A closed regular curve in space has total curvature $\int_0^L \kappa(s)\,ds \geq 2\pi$.