Normal curvature

First fundamental form.

$\bar{A} = \bar{E}_1 A^1 + \bar{E}_2 A^2$
$\bar{B} = \bar{E}_1 B^1 + \bar{E}_2 B^2$

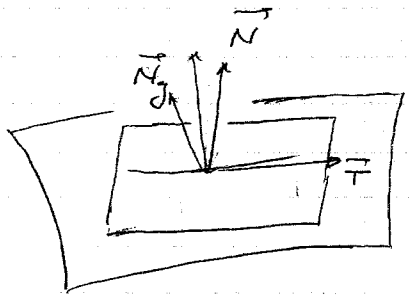
$\bar{A} \cdot \bar{B} = \sum_{i,j=1}^{2} g_{ij} A^i B^j$
$= E A^1 B^1 + F(A^1 B^2 + A^2 B^1) + G A^2 B^2$

Components A^i, B^i wrt $\bar{E}_1, \bar{E}_2$. $\bar{A}, \bar{B}$ tangent to surface.

Write as $I(\bar{A}, \bar{B})$.

Curve in a surface.

$S(\alpha(t))$, t any param



$\bar{N}_g = \bar{T} \times \bar{N}$

$\bar{V} = \frac{d}{dt} S(\alpha(t))$

s = arc length

$\bar{T} = \frac{d}{ds} S(\alpha(s))$

$\bar{V} = \bar{T} \frac{ds}{dt} = \sum_{i=1}^{2} \frac{\partial S}{\partial u^i}(\alpha(t)) \frac{d\alpha^i}{dt}(t) = \bar{E}_1 \frac{d\alpha^1}{dt} + \bar{E}_2 \frac{d\alpha^2}{dt}$.

$\left(\frac{ds}{dt}\right)^2 = \|\bar{V}\|^2 = \bar{V} \cdot \bar{V} = I(\bar{V}, \bar{V}) = \sum_{i,j=1}^{2} g_{ij} \frac{d\alpha^i}{dt} \frac{d\alpha^j}{dt}$

Acceleration vector.

$\frac{dV}{dt} = \frac{d}{dt}\left(\bar{T} \frac{ds}{dt}\right) = \left(\frac{d\bar{T}}{ds} \frac{ds}{dt}\right) \frac{ds}{dt} + \bar{T} \frac{d^2 s}{dt^2}$

$\frac{d\bar{T}}{ds} = \bar{N} \varkappa_n + \bar{N}_g \varkappa_g$

$= \underbrace{\bar{N} \varkappa_n \left(\frac{ds}{dt}\right)^2}_{\text{normal to surface.}} + \underbrace{\bar{T} \frac{d^2 s}{dt^2} + \bar{N}_g \varkappa_g \left(\frac{ds}{dt}\right)^2}_{\text{acceleration in surface (tangent to surface).}}$.

$\varkappa_n \left(\frac{ds}{dt}\right)^2 = \frac{d\vec{V}}{dt} \cdot \vec{N}$

Meusnier theorem.

Parametrize curve by arc length s:

$\varkappa_n = \frac{d\bar{T}}{ds} \cdot \bar{N}$

Serret-Frenet:

$\frac{d\vec{T}}{ds} = \left(\begin{matrix}\text{principal} \\ \text{normal}\end{matrix}\right) \varkappa$ ← space curvature

$\varphi = \angle\left(\vec{N}, \begin{matrix}\text{principle} \\ \text{normal}\end{matrix}\right)$: $\varkappa_n = \varkappa \cos\varphi$.

Acceleration – continued.

$$\vec{V} = \sum \vec{E}_i \frac{d\alpha^i}{dt}, \qquad \vec{E}_i = \frac{\partial S}{\partial u^i}, \text{ etc.}$$

$$\frac{d\vec{V}}{dt} = \sum \vec{E}_{ij} \frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt} + \sum \vec{E}_i \frac{d^2\alpha^i}{dt^2}$$

$$\vec{N} L_{ij} + \sum \vec{E}_r \Gamma^r_{ij}$$

$$= \vec{N}\left(\sum L_{ij}\frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt}\right) + \underbrace{\sum \vec{E}_h\left(\frac{d^2\alpha^h}{dt} + \sum \Gamma^h_{ij}\frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt}\right)}_{\text{tangent to surface.}}$$

$$\frac{d\vec{V}}{dt}\cdot\vec{N} = \sum_{i,j=1}^{2} L_{ij}\frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt}$$

Second fundamental form.

$$\left.\begin{aligned}\vec{A} &= \vec{E}_1 A^1 + \vec{E}_2 A^2\\ \vec{B} &= \vec{E}_1 B^1 + \vec{E}_2 B^2\end{aligned}\right\} \quad \text{II}(\vec{A},\vec{B}) = \sum_{i,j=1}^{2} L_{ij} A^i B^j$$

$$= L A^1 B^1 + M(A^1 B^2 + A^2 B^1) + N A^2 B^2.$$

$$L_{11} = L, \quad L_{12} = L_{21} = M, \quad L_{22} = N.$$

Normal Curvature.

$$\left.\begin{aligned}\kappa_n\left(\frac{ds}{dt}\right)^2 &= \frac{d\vec{V}}{dt}\cdot\vec{N}\\ \left(\frac{ds}{dt}\right)^2 &= \text{I}(\vec{V},\vec{V})\\ \frac{d\vec{V}}{dt}\cdot\vec{N} &= \text{II}(\vec{V},\vec{V})\end{aligned}\right\} \rightarrow \kappa_n(P) = \frac{\text{II}(\vec{V},\vec{V})}{\text{I}(\vec{V},\vec{V})}$$



Depends only on tangent direction at P.

$$\kappa_n(P) = \frac{L x^2 + 2M xy + N y^2}{E x^2 + 2F xy + G y^2}, \qquad \vec{E}_1 x + \vec{E}_2 y \text{ tangent vector at } P.$$

– sign is + or – according as principal normal and $\vec{N}$ are in same or opposite directions.