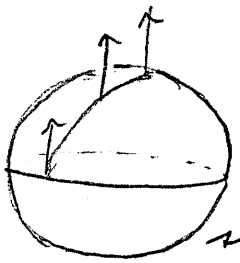
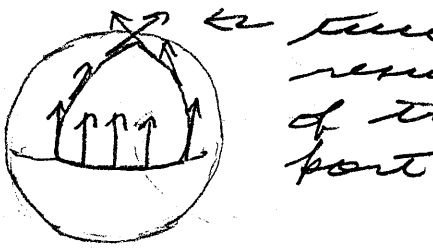
9. Abstract surfaces. Parallel transport

Motivation - surfaces in space

Cannot move a vector tangent to a surface merely by moving it in $\mathbb{R}^3$ - it fails to remain tangent to the surface.

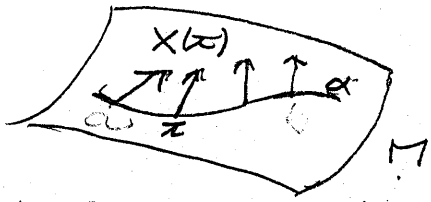


If one moves vector tangent to surface by taking "best possible" tangent vectors at each stage of the motion, one obtains different results depending on the path. The parallel transport to be defined is path dependent whenever the Gauss curvature is not identically zero.



Parallel transport

Let α be a curve on Σ. A vector field X on α is called parallel along α (in sense of Levi-Civita) if $\frac{DX}{dt} = 0$ along α.



If α and X are given in a chart by $\alpha: \begin{cases} u^1 = \alpha^1(t) \\ u^2 = \alpha^2(t) \end{cases}$, $X(t) = X^1(t) \frac{\partial}{\partial u^1}\Big|_{(\alpha^1(t), \alpha^2(t))} + X^2(t) \frac{\partial}{\partial u^2}\Big|_{(\alpha^1(t), \alpha^2(t))}$ (1)

then X is parallel along α exactly when

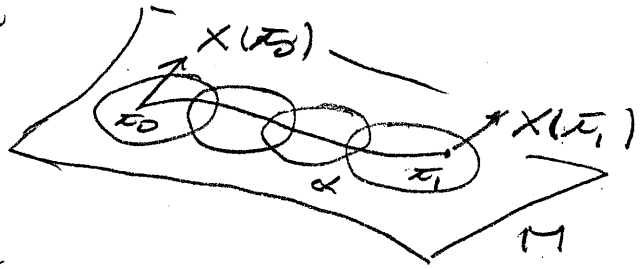
$$\frac{dX^a}{dt} + \Gamma^a_{ib} \frac{d\alpha^i}{dt} X^b = 0 .$$

$a = 1, 2$ and summed on $i, b = 1, 2$.

Remark For a surface S in space, X is parallel along $\sigma = S \circ \alpha$ exactly when $\frac{dX}{dt}$ is perpendicular to S at each point of σ.

To transport a vector X_0 tangent to M at $\alpha(t_0)$ to a vector tangent to M at $\alpha(t_1)$, one solves the initial value problem

$$\begin{cases} \frac{DX}{dt}(t) = 0 \\ X = X_0 \text{ at } t = t_0 \end{cases}$$



(D/dt is along α); the transported vector is $X(t_1)$.
In a local chart, the initial value problem is

$$\begin{cases} \frac{dX^a}{dt} + \sum_{i,b=1}^{2} \Gamma^a_{ib}(\alpha^1(t), \alpha^2(t)) \frac{d\alpha^i}{dt}(t) X^b = 0 \\ X^a = X^a_0 \text{ when } t = t_0 \end{cases} \qquad a = 1,2$$

Note the map $X(t_0) \mapsto X(t_1)$ is linear from vectors tangent to M at $\alpha(t_0)$ to vectors tangent to M at $\alpha(t_1)$.

Aside A curve is a geodesic if and only if its velocity vector is parallel along the curve.

Solutions to the ODE always exist locally. More than one chart of coordinates may be needed to cover the segment of the curve.

Note Parallel transport does not depend on the parametrization of α. For if $t = \varphi(t')$, $\frac{DX}{dt'} = \frac{DX}{dt}\frac{dt}{dt'}$.

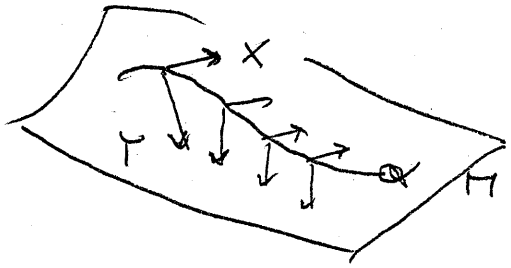
Effects of parallel transport.
If X and Y are parallel along α in M, then:

1) $X \circ Y$ = CONSTANT along α

For:
$$\frac{d}{dt}(X \circ Y) = \underbrace{\frac{DX}{dt} \circ Y}_{=0} + X \circ \frac{DY}{dt}$$

2) $\|X\|$ and $\|Y\|$ are constant along α

3) The angle between X and Y is constant along α. By
$$\cos \angle(X,Y) = \frac{X \circ Y}{\|X\| \|Y\|}$$

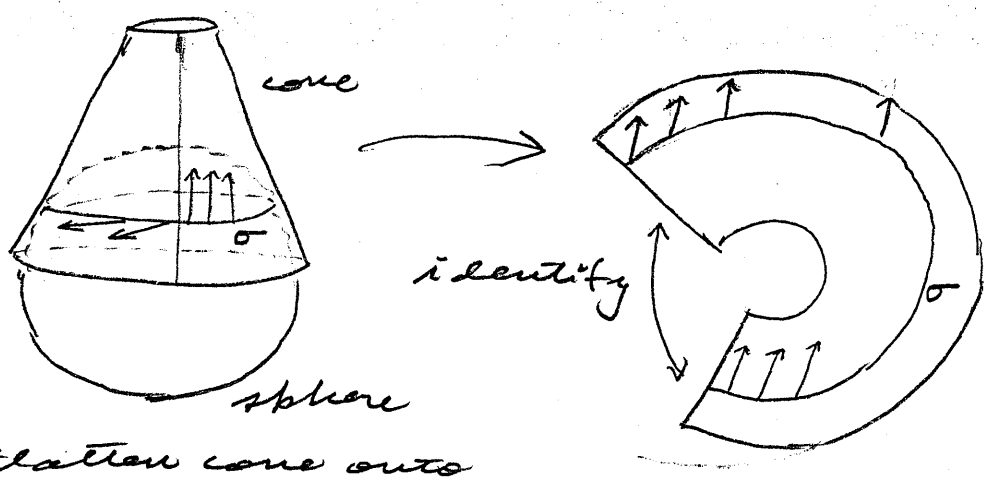


Note In the Euclidean plane: $ds^2 = du^2 + dv^2$, all the Christoffel symbols Γ^a_{bc} are zero. The intrinsic derivative D/dt coincides with the ordinary derivative d/dt.
$\frac{DX}{dt} = 0$ becomes $\frac{dX^a}{dt} = 0$, so $X^a = \text{const}$, $a = 1, 2$.
Parallel transport is the ordinary translation of vectors.

Parallel transport Levi-Civita's interpretation

If two regular surfaces in space are tangent along a curve σ, parallel transport of a vector along σ yields the same vector whether σ is regarded as lying in either surface.
For: "$\frac{dX}{dt}$ orthogonal to surface along σ" is the condition for parallel transport. And, the two surfaces share the same tangent plane at each point of σ.

Example



Flatten cone onto plane using the fact that parallel transport is intrinsic - a "bending invariant".

Flattened cone is part of Euclidean plane, so parallel transport is ordinary translation of vectors.

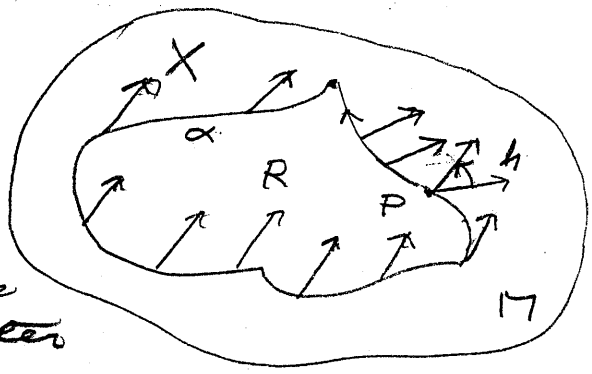
Levi-Civita's method

Given a curve σ on a surface S, the osculating developable - the envelope of tangent planes to S along σ - is a "flat surface" - Gauss curvature identically zero - that can be "rolled out" onto the plane and parallel transport effected by translation of vectors. This developable surface is tangent to S along σ, so one obtains the parallel transport on S.

The example above - standard in every exposition - can be misleading. The osculating developable is not even the use of the second fundamental form, not the first. Orthogonality with respect to these forms coincides in this example.

Curvature and parallel transport

Setting



M abstract surface
R oriented connected region in M
α oriented boundary of R, a piecewise regular curve
x arbitrary parameter
t ~~arbitrary~~ parameter
X a vector field parallel along α
M angle between X(a) at P = α(a) and X(b) at P = α(b)
- the angle through which X rotates when parallel transported around α
- Holonomy angle

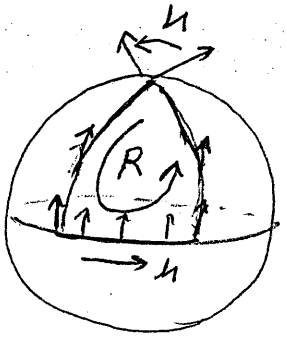
Remark R needs to be connected but need not be simply connected. The curve α needs to be connected. Orientability on an abstract surface is obtained by requiring $\frac{\partial(u,v)}{\partial(u',v')} > 0$ on the overlap of charts.

Theorem (Bonnet) $M = \int_R K\,dA$ where

K is the Gauss curvature on M
dA is the area element on M
$\int_R K\,dA$ is the total curvature of R

Note This holonomy angle is the same for all fields parallel along α.

Example



Π is sphere of radius a

$K = \frac{1}{a^2}$

μ angle in radians

area $R = \frac{\mu}{2\pi} \cdot \frac{1}{2} \cdot 4\pi a^2$

$$\int_R K \, dA = \frac{1}{a^2} \cdot \frac{\mu}{2\pi} \cdot \frac{1}{2} 4\pi a^2 = \mu$$

Proof - in several steps

1) We work in a single coördinate chart. We may assume that the coördinates are orthogonal. (Any non-vanishing field on an open set of Π has an orthogonal field. This field may be integrated to obtain coördinates.



2) Let

$$S_1(u^1,u^2) = \frac{\partial}{\partial u^1}\Big|_{(u^1,u^2)}$$

$$S_2(u^1,u^2) = \frac{\partial}{\partial u^2}\Big|_{(u^1,u^2)}$$

$F(u^1,u^2) = S_1 \circ S_2 = 0$

be the orthogonal coördinate fields.

Note These are not $\frac{\partial S}{\partial u^1}$ and $\frac{\partial S}{\partial u^2}$.

The fields

$$e_1(u^1,u^2) = S_1(u^1,u^2)\frac{1}{\sqrt{E(u^1,u^2)}}$$

$$e_2(u^1,u^2) = S_2(\dots)\frac{1}{\sqrt{G(\dots)}}$$

are orthonormal at each point.

We will need $\frac{De_1}{dt}$ and $\frac{De_2}{dt}$ along α.

3) From formulas p. 6-14 with $F=0$:

$$\Gamma_{11}^1 = \frac{1}{2}\frac{E_1}{E}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = \frac{1}{2}\frac{E_2}{E}, \quad \Gamma_{22}^1 = -\frac{1}{2}\frac{G_1}{E}$$

$$\Gamma_{11}^2 = -\frac{1}{2}\frac{E_2}{G}, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{2}\frac{G_1}{G}, \quad \Gamma_{22}^2 = \frac{1}{2}\frac{G_2}{G}$$

Now, $\left(\frac{DX}{dt}\right)^a = \frac{dX^a}{dt} + \Gamma^a_{ij}\dot{\alpha}^i X^b$.

The field S_1 has $X^1 = 1$ and $X^2 = 0$, so

$$\left(\frac{DS_1}{dt}\right)^1 = \frac{d1}{dt} + \Gamma^1_{11}\dot{\alpha}^1 1 + \Gamma^1_{12}\dot{\alpha}^1 0 + \Gamma^1_{21}\dot{\alpha}^2 1 + \Gamma^1_{22}\dot{\alpha}^2 0$$

$$= \frac{1}{2}\frac{E_1}{E}\dot{\alpha}^1 + \frac{1}{2}\frac{E_2}{E}\dot{\alpha}^2$$

Similarly one obtains

$$\left(\frac{DS_1}{dt}\right)^2 = -\frac{1}{2}\frac{E_2}{G}\dot{\alpha}^1 + \frac{1}{2}\frac{G_1}{G}\dot{\alpha}^2$$

$$\left(\frac{DS_2}{dt}\right)^1 = \frac{1}{2}\frac{E_2}{E}\dot{\alpha}^1 - \frac{1}{2}\frac{G_1}{E}\dot{\alpha}^2$$

$$\left(\frac{DS_2}{dt}\right)^2 = \frac{1}{2}\frac{G_1}{G}\dot{\alpha}^1 + \frac{1}{2}\frac{G_2}{G}\dot{\alpha}^2$$

3) One can now compute

$$\frac{De_1}{dt} = \frac{D}{dt}\left(S_1\frac{1}{\sqrt{E}}\right) = \frac{DS_1}{dt}\frac{1}{\sqrt{E}} + S_1\frac{d}{dt}\left(\frac{1}{\sqrt{E}}\right)$$

$$\left(\frac{\partial}{\partial u^1}\frac{1}{\sqrt{E}}\right)\dot{\alpha}^1 + \left(\frac{\partial}{\partial u^2}\frac{1}{\sqrt{E}}\right)\dot{\alpha}^2$$

$$= S_1\underbrace{\left(\frac{1}{2}\frac{E_1}{E}\dot{\alpha}^1 + \frac{1}{2}\frac{E_2}{E}\dot{\alpha}^2\right)}_{\left(\frac{DS_1}{dt}\right)^1}\frac{1}{\sqrt{E}} + S_2\underbrace{\left(-\frac{1}{2}\frac{E_2}{G}\dot{\alpha}^1 + \frac{1}{2}\frac{G_1}{G}\dot{\alpha}^2\right)}_{\left(\frac{DS_1}{dt}\right)^2}\frac{1}{\sqrt{E}}$$

$$+ S_1\left(-\frac{1}{2}\frac{E_1}{E^{3/2}}\dot{\alpha}^1 - \frac{1}{2}\frac{E_2}{E^{3/2}}\dot{\alpha}^2\right)$$

$$= \underbrace{S_2\frac{1}{\sqrt{G}}}_{e_2}\left(-\frac{1}{2}\frac{E_2}{\sqrt{EG}}\dot{\alpha}^1 + \frac{1}{2}\frac{G_1}{\sqrt{EG}}\dot{\alpha}^2\right)$$

Thus $\frac{De_1}{dt} = e_2\,\omega(t)$ with

$$\omega(t) = \frac{1}{2}\frac{1}{\sqrt{EG}}\left(-E_2\dot{\alpha}^1 + G_1\dot{\alpha}^2\right)$$

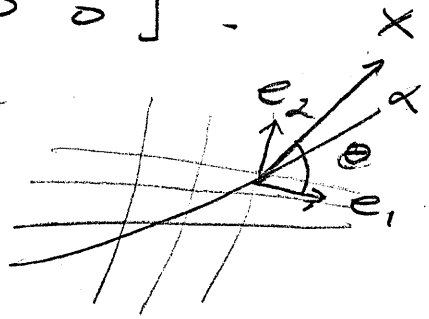
$\leftarrow E(\alpha^1(t), \alpha^2(t))$, etc.

A similar calculation yields

$$\frac{De_2}{dt} = -e_1\,\omega(t), \quad \text{same } \omega(t).$$

The angle $\omega(t)$ is the rate of rotation of the frame e_1, e_2 along α:

$$\frac{D}{dt}[e_1\ e_2] = [e_1\ e_2]\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}.$$



4) Now examine parallel transport in terms of the frame e_1, e_2.

Let X be a vector field along α and

$\theta(t)$ = angle $e_1(\alpha^1(t), \alpha^2(t))$ to $X(t)$

Write

$$X(t) = \underbrace{(e_1 \cos\theta + e_2 \sin\theta)}_{\text{unit vector}} A$$

$e_1(\alpha^1(t), \alpha^2(t))$ | $\theta(t)$ | amplitude $A(t) = \|X(t)\|$

Then

$$\frac{DX}{dt} = \frac{D}{dt}(e_1\cos\theta + e_2\sin\theta)A$$

$$+ (e_1\cos\theta + e_2\sin\theta)\frac{dA}{dt}$$

$$= \Big(e_2\omega\cos\theta + e_1(-\sin\theta)\frac{d\theta}{dt}$$

$$+ (-e_1\omega)\sin\theta + e_2\cos\theta\frac{d\theta}{dt}\Big)A$$

$$+ (e_1\cos\theta + e_2\sin\theta)\frac{dA}{dt}$$

$$= (-e_1\sin\theta + e_2\cos\theta)\Big(\omega + \frac{d\theta}{dt}\Big)A$$

e_2: ORTHONORMAL TO VECTORS

$$+ (e_1\cos\theta + e_2\sin\theta)\frac{dA}{dt}.$$

Thus, X is parallel along α is equivalent to

$$\frac{DX}{dt} = 0 \quad \text{or} \quad \omega + \frac{d\theta}{dt} = 0 \quad \text{along } \alpha.$$

(A = CONSTANT)

5) We can now calculate the holonomy angle for a vector field X parallel along α. It is:

$$h = \int_{\substack{\text{closed} \\ \text{curve } \alpha}} d\theta = \int_a^b \frac{d\theta}{dt}\, dt = \int_a^b -\omega\, dt$$

$$= -\frac{1}{2}\int_a^b \frac{1}{\sqrt{EG}}\left(-E_2 \frac{d\alpha^1}{dt} + G_1 \frac{d\alpha^2}{dt}\right) dt$$

$$= -\frac{1}{2}\int_{\substack{\text{closed} \\ \text{curve } \alpha}} -\frac{E_2}{\sqrt{EG}}\, du^1 + \frac{G_1}{\sqrt{EG}}\, du^2$$

Use Green's theorem in plane:

$$\int_{\text{boundary } R} P\, du + Q\, dv = \iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\, dv .$$

Then

$$h = -\frac{1}{2}\iint_R \left(\frac{\partial}{\partial u^1}\left(\frac{G_1}{\sqrt{EG}}\right) - \frac{\partial}{\partial u^2}\left(-\frac{E_2}{\sqrt{EG}}\right)\right) du^1 du^2 .$$

Now, Problem 5.5:

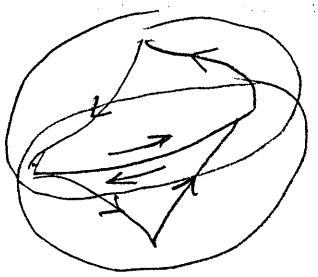
$$K = -\frac{1}{\sqrt{EG}}\left(\frac{\partial}{\partial u^1}\underbrace{\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u^1}\right)}_{\frac{1}{2}\frac{G_1}{\sqrt{EG}}} + \frac{\partial}{\partial u^2}\underbrace{\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial u^2}\right)}_{\frac{1}{2}\frac{E_2}{\sqrt{EG}}}\right)$$

and $dA = \sqrt{EG}\, du^1 du^2$ (area, $F=0$), so

$$h = \iint_R K\, dA .$$

6) When the region R is not covered by a single chart, use the fact that

the integral is additive over regions and appropriate line integrals cancel.



R two charts

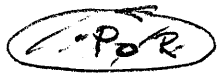
R' And, the quantities involved do not depend on choice of coordinates. QED

Consequence Parallel transport is independent of paths if and only if the Gaussian curvature is identically zero:

For, $K = 0$ implies $\mu = \int K\,dA = 0$

If $K(P_0) \neq 0$, find a region R about P_0 where K does not change sign. Then $\mu = \int K\,dA \neq 0$, so transport around the boundary of R has $\mu \neq 0$.





Separate the boundary of R into two curves to obtain different parallel transports between two points.



In the Euclidean plane, $K = 0$ identically and parallel transport is path independent.