Lecture 8. Abstract surfaces. Global notions.

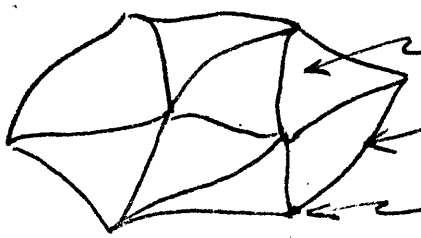
This lecture summarizes without proof several facts from the topology of two-dimensional manifolds.

Remark Two two-dimensional manifolds M and M' are topologically equivalent if there is a continuous map from M onto M' which is one-to-one with continuous inverse. Usually the map can be taken to be differentiable with differentiable inverse. The notions intrinsic to M are unchanged if the manifold is replaced by one topologically equivalent to it.

Triangulations

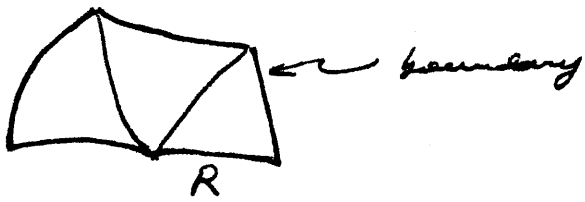
Every two-dimensional manifold is triangulable i.e. can be built up of triangles matched along their edges and vertices.

N.B. The edges of the triangles are "curved lines"



Note: There are many triangulations of the same manifold

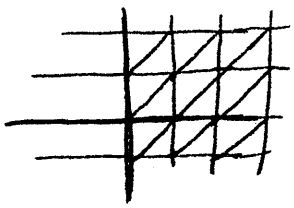
Note: A region R of M can be built up of triangles also. The boundary of R consists of the edges common to only one triangle.



Closed or compact manifold: M has a triangulation consisting of finitely many triangles.

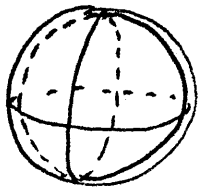
Open or non-compact manifold: M has no triangulation consisting of finitely many triangles.

Example 1



Plane. Open surface.

Example 2

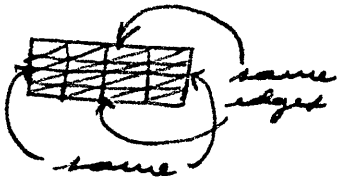


Sphere closed surface.

Example 3



Torus. Closed surface.

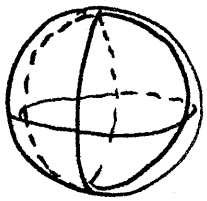


Euler characteristic

If M is a ~~surface~~ closed manifold with a triangulation having F faces, E edges, V vertices, the Euler characteristic of M is $\chi(M) = F - E + V$.

N.B. This does not depend on the triangulation.

Example



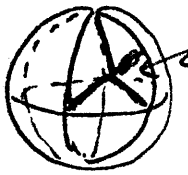
Sphere

$F = 8$

$E = 12$

$V = 6$

$\chi = 8 - 12 + 6 = 2.$



reflect in each octant

$F = 24$

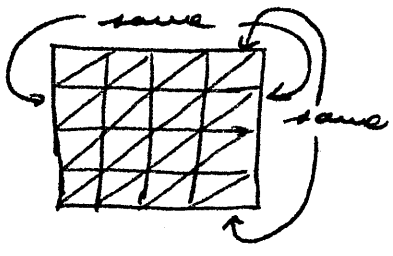
$E = 36$

$V = 14$

$\chi = 24 - 36 + 14 = 2.$

Note: For a triangulated region R, $\chi(R) = F - E + V$ in the same way.

Example



Torus

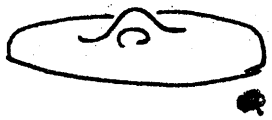
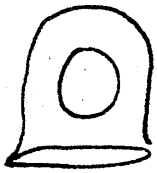
$F = 32$

$E = 48$

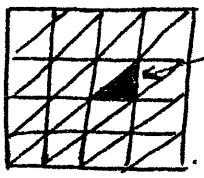
$V = 16$

$\chi = 0.$

Example


or

or


Torus with a disc removed.


remove

$F = 31$

$E = 48$

$V = 16$

$\chi = 31 - 48 + 16 = -1.$

Orientability

A triangle can be oriented many ways with its boundary:

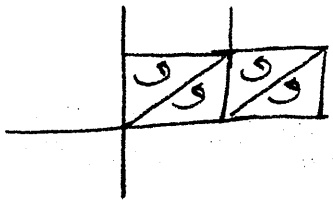


If the surface or region can be made up of oriented triangles coherently oriented along the edges, the surface or region is orientable; non-orientable if not.



Coherent orientations of the triangles.

Example



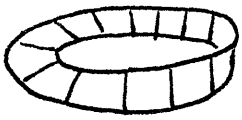
Plane; orientable.

Example

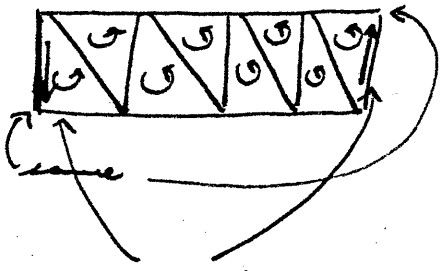


Sphere orientable.

Example



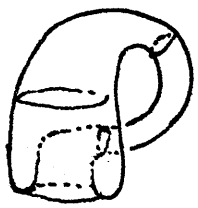
Möbius band; non-orientable.



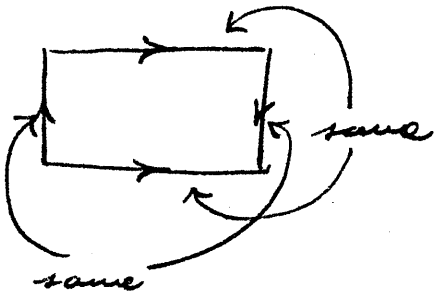
"One-sided surface"

not coherent - directions should be opposite but are like.

Example



Klein bottle; non-orientable.



"One-sided surface"

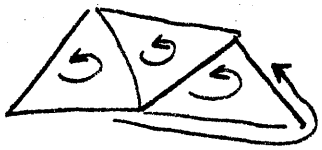
N.B. This example cannot exist as a surface in space.

Remark A two-dimensional manifold M is orientable if and only if it has a cover by coordinate patches $\{U\}$ such that on $U \cap U'$

the Jacobian $\frac{\partial(u', v')}{\partial(u, v)}$ is > 0.

Remark A surface in space S (without self intersections (an embedded or smooth surface)) is orientable if and only if there is a ~~continuous~~ differentiable normal vector field N over all of S. $N = \frac{\frac{\partial \vec{S}}{\partial u} \times \frac{\partial \vec{S}}{\partial v}}{\left| \frac{\partial \vec{S}}{\partial u} \times \frac{\partial \vec{S}}{\partial v} \right|}$ locally.

Note: If a region is oriented, its boundary becomes oriented also.



Genus

M two dimensional closed and orientable.

$\chi(M)$ = even integer

$= 2 - 2g$

g = genus of M

g is the number of "handles" to be added to a sphere to obtain M.

Examples

1) 

Sphere.

$\chi = 2$

$g = 0$.

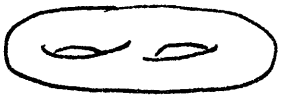
2) 

Torus.

or 

$\chi = 0$

$g = 1$

3) 

or 

$\chi = -2$

$g = 2$

4) 

or 

$\chi = -4$

$g = 3$

Etc.

Remark. There is a genus and standard description for non-orientable closed ~~surfaces~~ manifolds also.

Fact. Two ~~surfaces~~ two-dimensional closed manifolds are topologically equivalent if and only if:

i) both are orientable or not;

ii) both have the same genus or Euler characteristic.

One can also "classify", in the above sense, open two-dimensional manifolds.

Remarks on non-orientable surfaces and general regions.

1) R any region on a manifold,

R + handle is :



Remove triangle from both and join along boundary.

$\chi(R + \text{handle}) = \chi(R) - 2.$

2) Any closed surface may be built out of:

sphere ($\chi = 2$), ~~[illegible]~~ projective plane ($\chi = 1$), Klein bottle ($\chi = 0$) by adding handles.

$\chi = \chi\left(\left(\begin{matrix}\text{sphere}\\ \text{proj. plane}\\ \text{Klein bottle}\end{matrix}\right) + g \text{ handles}\right)$

$= (2, 1, \text{or } 0) - 2g.$

3) Cutting a disc out of a region reduces it Euler charac. by 1. If R is a sphere with g handles and r boundary components,

$\chi(R) = 2 - 2g - r.$

4) A two-manifold is "classified" if we know

i) if it is orientable or not;

ii) its Euler characteristic;

iii) the number of boundary components.