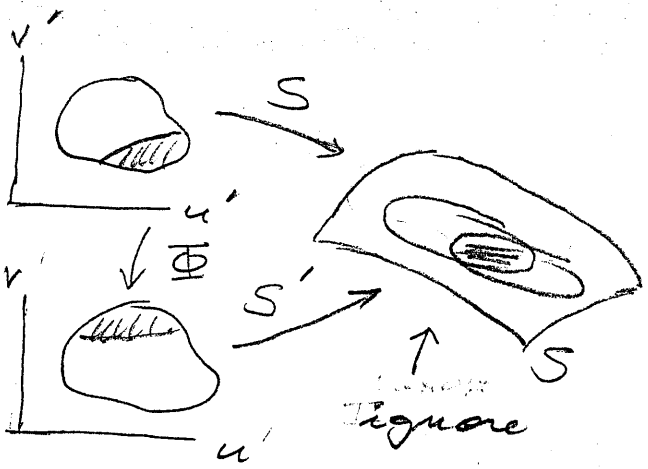
7. Abstract surfaces. Local notions

Motivation

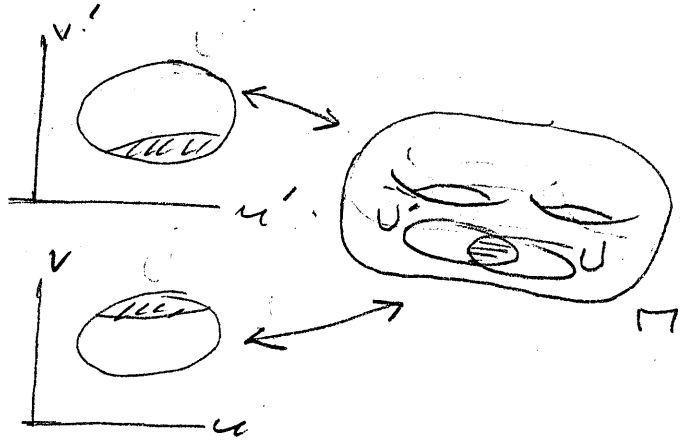
Quantities involving ds^2 only which can be computed from any choice of parameters are intrinsic and one can ignore the surface being in Euclidean space.



Two-dimensional manifolds

A two-manifold M is a Hausdorff topological space with a family of open sets $\{U\}$ called an atlas satisfying:

$\begin{cases} u = u(u', v') \\ v = v(u', v') \end{cases}$



0) $\{U\}$ covers M

1) Each U is homeomorphic with an open set in $\mathbb{R}^2$ via coördinates $u = u'$, $v = u^2$. The U are coördinate patches or coördinate neighborhoods called charts.

2) If $U \cap U' \neq \emptyset$ in M, then
$\begin{cases} u = u(u', v') \\ v = v(u', v') \end{cases}$ on the images of the overlap in $\mathbb{R}^2$,
with the functions of class C^3, invertable, and $\dfrac{\partial(u, v)}{\partial(u', v')} \neq 0$.

3) M is connected.

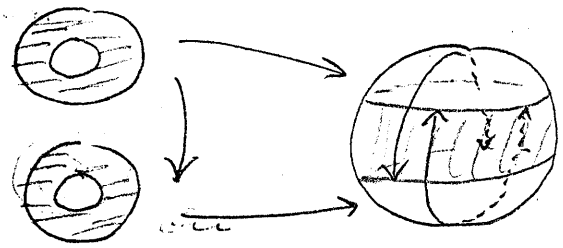
4) Any open set U with functions u & v satisfying 1) and 2) is already in the atlas. This allows one to introduce new coordinate patches or take parts

of old ones without changing the manifold. It suffices to work with a subfamily of charts.

5) M is countable: a countable subfamily of $\{U\}$ covers M. This prevents the manifold from becoming too "large".

Example

The 2-sphere. Charts are stereographic projection.



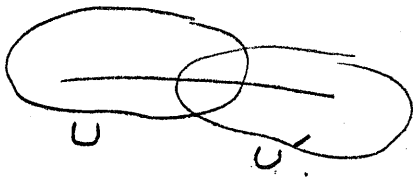
Problem (7.1) Let M be the 2-sphere in $\mathbb{E}^3$. Let $\sigma : M \setminus \{\text{north pole}\} \to \mathbb{R}^2$ be stereographic projection from the north pole to the plane tangent at the south pole, and let $\tau : M \setminus \{\text{south pole}\} \to \mathbb{R}^2$ be the map with poles interchanged. Find a formula for $\sigma \circ \tau^{-1} : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$

Curves - are described in terms of the coördinates in the charts:

$u = u(t)$, $v = v(t)$

For a second chart:

$u' = u'(t)$, $v' = v'(t)$.



On the overlap,

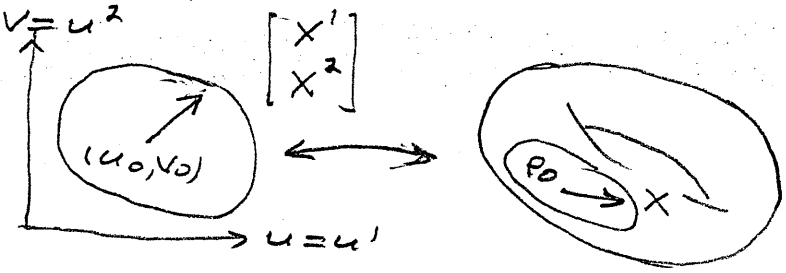
$$\begin{cases} u(t) = u(u'(t), v'(t)) \\ v(t) = v(u'(t), v'(t)). \end{cases}$$

A curve is as smooth as the differentiability of $u(t)$ & $v(t)$ allows. This is unchanged by a change of coördinates which is sufficiently differentiable.

Remark M connected means: any two points of M can be joined by a continuous curve.

Tangent vectors - at a point.

Give components X^1 and X^2 of X in each chart.



Write

$$X = X^1 \frac{\partial}{\partial u}\Big|_{(u_0,v_0)} + X^2 \frac{\partial}{\partial v}\Big|_{(u_0,v_0)}$$

to remind us of

$$X = \frac{\partial S}{\partial u}(u_0,v_0)\, X^1 + \frac{\partial S}{\partial v}(u_0,v_0)\, X^2$$

for surfaces in space. For a second chart

$$\begin{cases} \dfrac{\partial}{\partial u'} = \dfrac{\partial u}{\partial u'}\dfrac{\partial}{\partial u} + \dfrac{\partial v}{\partial u'}\dfrac{\partial}{\partial v} \\ \dfrac{\partial}{\partial v'} = \dfrac{\partial u}{\partial v'}\dfrac{\partial}{\partial u} + \dfrac{\partial v}{\partial v'}\dfrac{\partial}{\partial v} \end{cases}$$

- by chain rule
- evaluated at appropriate points.

or - as change of basis of tangent vectors:

$$\begin{bmatrix} \frac{\partial}{\partial u'}, & \frac{\partial}{\partial v'} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial u} & \frac{\partial}{\partial v} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial u'}, & \frac{\partial u}{\partial v'} \\ \frac{\partial v}{\partial u'}, & \frac{\partial v}{\partial v'} \end{bmatrix} \leftarrow \text{at } (u_0', v_0')$$

For $X = X'^1 \frac{\partial}{\partial u'} + X'^2 \frac{\partial}{\partial v'}$, we have

$$\begin{bmatrix} X'^1 \\ X'^2 \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial u'}, & \frac{\partial u}{\partial v'} \\ \frac{\partial v}{\partial u'}, & \frac{\partial v}{\partial v'} \end{bmatrix}^{-1} \begin{bmatrix} X^1 \\ X^2 \end{bmatrix} \quad \text{at } P_0 \ldots$$

Cf Problems 3.3 and 4.2.

Remark One also writes

$$\frac{\partial}{\partial u'^r} = \frac{\partial u^i}{\partial u'^r}\frac{\partial}{\partial u^i}, \quad X'^r = \frac{\partial u'^r}{\partial u^i} X^i$$

- summed on repeated indices, so

$$X = X^i \frac{\partial}{\partial u^i} = X'^r \frac{\partial}{\partial u'^r}.$$

Velocity vector to a curve at a point

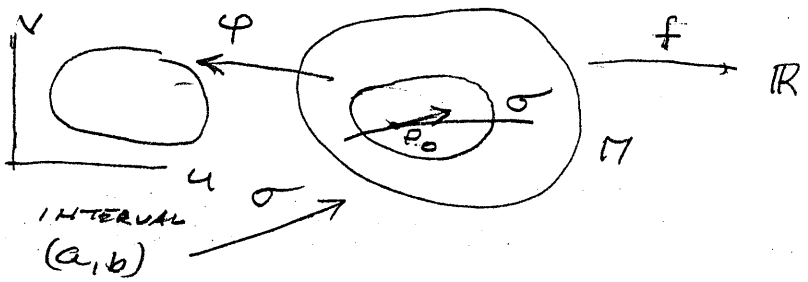
If a curve σ on M is given by $\begin{cases} u = u(t) \\ v = v(t) \end{cases}$ in a chart with coordinates (u,v), we write

$$\frac{d\sigma}{dt}(t_0) = \frac{du}{dt}(t_0)\frac{\partial}{\partial u}\Big|_{(u(t_0),v(t_0))} + \frac{dv}{dt}(t_0)\frac{\partial}{\partial v}\Big|_{(-,-)}$$

for the velocity vector of σ at $\sigma(t_0)$. This respects changes of coördinates.

Problem (7.2)

The coördinates u & v are actually a function φ on M to $\mathbb{R}^2$.



If $f : M \to \mathbb{R}$ is a real-valued function on M, writing $f(u,v)$ really means $(f \circ \varphi^{-1})(u,v)$.

a) Let X be a vector tangent to M at P_0, and let σ be any curve such that $\sigma(t_0) = P_0$ and $\frac{d\sigma}{dt}(t_0) = X$. Using $f \circ \sigma = (f \circ \varphi^{-1}) \circ (\varphi \circ \sigma)$, convince yourself that X (at P_0) is the "directional derivative" of f along any such σ at t_0:

$$\frac{d}{dt} f(\sigma(t))\Big|_{t=t_0} = \frac{\partial f}{\partial u}(u(t_0), v(t_0))\frac{du}{dt}(t_0) + \frac{\partial f}{\partial v}(-,-)\frac{dv}{dt}(t_0)$$

This is written Xf.

b) Show $f \mapsto Xf$ is linear and

$$X(fg) = (Xf)g + f(Xg).$$

Thus: Vectors tangent to an abstract manifold M can be obtained as operations called derivations on the real functions on M.

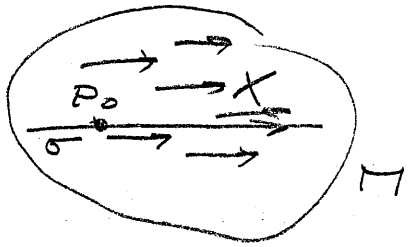
Vector fields A *vector field* on M is a map $p \mapsto X_p = (\text{a vector tangent to } M \text{ at } p)$.
In coordinates,

$$X = X^1(u,v)\frac{\partial}{\partial u} + X^2(u,v)\frac{\partial}{\partial v}$$

with components X^1 & X^2 sufficiently smooth.

Differential equations A first-order ODE is a vector field X on M. An *integral curve* through a point p_0 is a curve $\sigma(t)$ satisfying

$$\begin{cases} \frac{d\sigma}{dt}(t) = X_{\sigma(t)} \\ \sigma(t_0) = p_0 \end{cases}$$



In a chart with coordinates (u,v), this is

$$\begin{cases} \frac{du}{dt}(t) = X^1(u(t), v(t)) \\ \frac{dv}{dt}(t) = X^2(u(t), v(t)) \\ u(t_0) = u_0 \\ v(t_0) = v_0 \end{cases}$$

DIFFERENTIAL EQUATION

INITIAL CONDITIONS (u_0, v_0 coords of p_0)

Abstract surfaces
— or *two-dimensional Riemannian manifolds*

Equip a two-manifold M with a smooth field of positive definite quadratic forms. This is called the *metric* (FIRST FUND. FORM). If X and Y are tangent vectors of M at p, then $X \circ Y$ is the product in the metric.
In coordinates (u,v):

$$E = \frac{\partial}{\partial u} \circ \frac{\partial}{\partial u}, \quad F = \frac{\partial}{\partial u} \circ \frac{\partial}{\partial v}, \quad G = \frac{\partial}{\partial v} \circ \frac{\partial}{\partial v} \text{ at } p$$

For $X = X^1\frac{\partial}{\partial u} + X^2\frac{\partial}{\partial v}$ at p, we have

$$X \circ X = E(X^1)^2 + 2FX^1X^2 + G(X^2)^2.$$

or $ds^2 = \sum_{i,j=1}^{2} g_{ij}(u^1,u^2)\,du^i\,du^j$.

This respects change of coördinates.

Quantities that respect change of coördinates - "covariant" - are geometric. Especially, - quantities depending only on the metric are geometric.

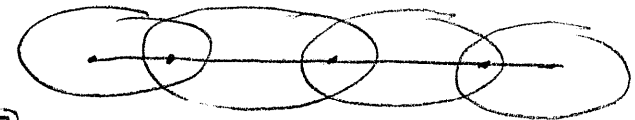
Geometric entities - cf. "intrinsic".

At one point:

- inner product $X \circ Y$ - X & Y vectors at same point
- length $\|X\| = \sqrt{X \circ X}$ - metric positive definite
- angle $\cos \angle(X,Y) = \dfrac{X \circ Y}{\|X\|\,\|Y\|}$
- orthogonality $X \circ Y = 0$

At several points - integration

- length of a curve $\sigma(t)$: $\displaystyle\int_a^b \left\| \frac{d\sigma}{dt}(t) \right\| dt$.



It may take several charts to cover the curve. In each chart the length of the segment is

$$\int \sqrt{E(u(t),v(t))\left(\frac{du}{dt}(t)\right)^2 + \cdots}\; dt$$

One adds the lengths of the segments.

- orthogonal curves $\dfrac{d\sigma}{dt} \circ \dfrac{d\tau}{dt} = 0$
- orthogonal coördinates - $\dfrac{\partial}{\partial u} \circ \dfrac{\partial}{\partial v} = 0$ or $F = 0$ in a chart.
- area element dA - in a chart $dA = \sqrt{EG - F^2}\,du\,dv$

At several points - differentiation

- Gauss curvature - Theorema Egregium
- intrinsic or covariant derivative along a curve $u^1(t), u^2(t)$:

$$\frac{DX}{dt}(t) = \left(\frac{DX}{dt}\right)^1(t)\frac{\partial}{\partial u^1}\Big|_{(u^1(t),u^2(t))} + \left(\frac{DX}{dt}\right)^2(t)\frac{\partial}{\partial u^2}\Big|_{(u^1(t),u^2(t))}$$

$$\left(\frac{DX}{dt}\right)^a(t) = \frac{dX^a}{dt}(t) + \sum_{i,b}\Gamma^a_{ib}(u^1(t),u^2(t))\frac{du^i}{dt}(t)X^b(t)$$

- geodesic curvature

$$\begin{cases}\frac{DT}{ds} = U\times_g & T = \frac{d\sigma}{ds}\\ \frac{DU}{ds} = -T\times_g & T, U \text{ orthonormal along } \sigma\end{cases}$$

- geodesics

$\times_g = 0$ or $\frac{D}{ds}\frac{d\sigma}{ds} = 0$.

- geodesics as minimizing arc length.

- variation of $L(\varepsilon) = \int_a^b \left\|\frac{D\sigma}{\partial s}(s,\varepsilon)\right\| ds$.

Note - need to use $\frac{D}{\partial\varepsilon}$ for the variation. And $\frac{D}{\partial\varepsilon}\frac{\partial\sigma}{\partial s} = \frac{D}{\partial s}\frac{\partial\sigma}{\partial\varepsilon}$. The proof is as before.

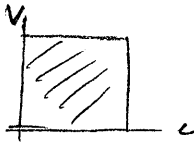
One obtains

$$\frac{dL}{d\varepsilon}(0) = \left[\frac{\partial\sigma}{\partial\varepsilon}(s,0)\circ\frac{\partial\sigma}{\partial s}(s,0)\right]_{s=0}^{s=l} - \int_0^l \frac{\partial\sigma}{\partial\varepsilon}(s,0)\circ\frac{D}{\partial s}\frac{\partial\sigma}{\partial s}(s,0)\,ds$$

- and use variations of segments of the curve to obtain $\times_g = 0$.

Example The Clifford torus.

$$S(u,v) = \begin{bmatrix}\cos u\\ \sin u\\ \cos v\\ \sin v\end{bmatrix},$$



$0\le u<2\pi$, $0\le v<2\pi$, $S \to E^4$

Take metric induced by Euclidean metric of $\mathbb{E}^4$.

$$\frac{\partial S}{\partial u} = \begin{bmatrix} -\sin u \\ \cos u \\ 0 \\ 0 \end{bmatrix}, \quad \frac{\partial S}{\partial v} = \begin{bmatrix} 0 \\ 0 \\ -\sin v \\ \cos v \end{bmatrix},$$

so

$$E = \frac{\partial S}{\partial u} \cdot \frac{\partial S}{\partial u} = 1, \quad F = \frac{\partial S}{\partial u} \cdot \frac{\partial S}{\partial v} = 0, \quad G = \frac{\partial S}{\partial v} \cdot \frac{\partial S}{\partial v} = 1$$

Thus

$ds^2 = du^2 + dv^2$ and $K = 0$ identically.

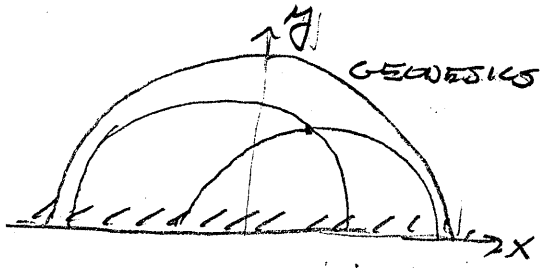
The surface lies in the three-sphere $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 2$ and divides the 3-sphere into two congruent parts.

Remark This torus cannot be isometrically embedded in three-space $\mathbb{E}^3$, since a closed surface in three space must have points where the Gaussian curvature is positive.

Example The Poincaré upper half-plane.

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

on $y > 0$. One chart suffices. One check $K = -1$ identically.

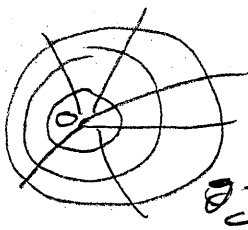


Note This is one "model" of the "non" Euclidean plane.

Theorem of Hilbert. The abstract surface above can not be isometrically embedded in three-dimensional Euclidean space.

Problem (7.3) Show that the geodesics in the Poincaré upper half-plane are semi-circles and lines orthogonal (Euclidean) to the x-axis.

Geodesic polar coördinates – Abstract surfaces



Fix a point O (origin) and a direction ($\theta=0$) from which to measure angles at O. Obtain **geodesic polar coördinates** (r,θ) near O by measuring the distance r along the geodesic through O having angle θ. In these coördinates

$$ds^2 = E(r,\theta)\,dr^2 + 2F(r,\theta)\,dr\,d\theta + G(r,\theta)\,d\theta^2.$$

It is, in fact, cleaner than this.

a) On a geodesic $\theta=\text{const}$, $ds=dr$, so $E(r,\theta)=1$ is constant.

b) Examine the equations for a geodesic:

$$\frac{d^2r}{ds^2} + \Gamma^1_{11}\left(\frac{dr}{ds}\right)^2 + 2\Gamma^1_{12}\frac{dr}{ds}\frac{d\theta}{ds} + \Gamma^1_{22}\left(\frac{d\theta}{ds}\right)^2 = 0$$

$$\frac{d^2\theta}{ds^2} + \Gamma^2_{11}\left(\frac{dr}{ds}\right)^2 + 2\Gamma^2_{12}\frac{dr}{ds}\frac{d\theta}{ds} + \Gamma^2_{22}\left(\frac{d\theta}{ds}\right)^2 = 0$$

The radial lines $r=s$, $\theta=\text{const}$, are geodesics, so

$$\Gamma^1_{11} = 0 \quad \text{and} \quad \Gamma^2_{11} = 0.$$

Now

$$\Gamma^1_{11} = \frac{GE_1 - 2FF_1 + FE_2}{2(EG-F^2)}, \text{ so } \frac{-FF_1}{G-F^2} = 0$$

$$\Gamma^2_{11} = \frac{-FE_1 + 2EF_1 - EE_2}{2(EG-F^2)}, \text{ so } \frac{F_1}{G-F^2} = 0.$$

Thus, $F_1 = \frac{\partial F}{\partial r}$ is zero. $F = F(\theta)$ depends on θ only.

c) Observe,

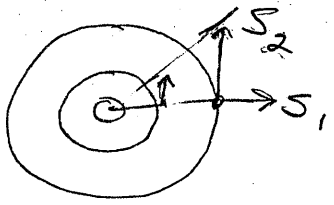
$$S_1 = \frac{\partial}{\partial r}\Big|_{(r,\theta)}$$ is unit vector tangent to radial geodesics

$S_2 = \frac{\partial}{\partial \theta}\Big|_{(r,\theta)}$ is velocity vector tangent to geodesic circle

As a velocity vector,

$\lim_{r\to 0} S_2 = 0$, since

$0 \le \theta \le 2\pi$ and lengths go to zero as $r \to 0$.



Thus $F(r,\theta) = S_1 \cdot S_2 \to 0$ as $r \to 0$

Since F does not depend on r,

$F(r,\theta) = 0$ all r, θ.

Consequence: In geodesic polar coordinates,

$ds^2 = dr^2 + G(r,\theta)\, d\theta^2$.

Curvature We need to analyze G near $r = 0$.

d) On a geodesic circle of radius r,

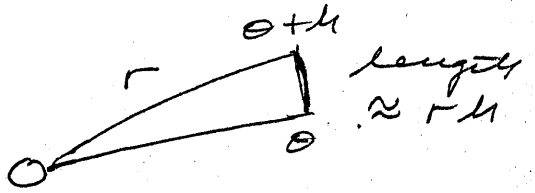
$ds = \sqrt{G(r,\theta)}\, d\theta$, so:

$$\begin{pmatrix}\text{length of geod. circle of radius } r\end{pmatrix} = \int_0^{2\pi} \sqrt{G(r,\theta)}\, d\theta$$

As $r \to 0$, this length $\to 0$, so

$$\lim_{r\to 0} \sqrt{G(r,\theta)} = 0$$

e) On a geodesic circle:



$$\begin{pmatrix}\text{length of geod. arc}\end{pmatrix} = \int_\theta^{\theta+h} \sqrt{G(r,x)}\, dx$$

$$= \sqrt{G(r,\theta+\eta)}\, h \quad \text{- by integral mean value theorem}$$
$$0<\eta<h$$

$$= \left(\sqrt{G(r,\theta+\eta)} - \underbrace{\sqrt{G(0,\theta+\eta)}}_{=0}\right) h$$

$$= \frac{\partial \sqrt{G}}{\partial r}(\xi, \theta+\eta)\, r h \quad \text{- by differential mean value theorem}$$
$$0<\xi<r$$

Note η & ξ depend on r, θ.

The value of this integral should be approximately $r+h$, so

$$\lim_{\substack{r\to 0\\ h\to 0}} \frac{1}{r+h}\int_{\theta}^{\theta+h} \sqrt{G(r,t)}\,dt = 1$$

or

$$\lim_{r\to 0} \frac{\partial\sqrt{G}}{\partial r}(r,\theta) = 1.$$

f) Formula for curvature (Prob. 5.5):

$$K(r,\theta) = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial r^2}.$$

Set $K_0 = K(0,\theta)$ (same for all θ).

g) To establish a power series for G:

$$\sqrt{G(r,\theta)} = \sqrt{G(0,\theta)} + \frac{\partial\sqrt{G}}{\partial r}(0,\theta)\,r + \frac{1}{2!}\frac{\partial^2\sqrt{G}}{\partial r^2}(0,\theta)\,r^2 + \frac{1}{6}\frac{\partial^3\sqrt{G}}{\partial r^3}(0,\theta)\,r^3 + O(r^4)$$

Note that θ enters the estimate in $O(r^4)$.

Now

$$\frac{\partial^2\sqrt{G}}{\partial r^2}(r,\theta) = -K(r,\theta)\sqrt{G}(r,\theta)$$

and

$$\frac{\partial^3\sqrt{G}}{\partial r^3} = -\frac{\partial K}{\partial r}\sqrt{G} - K\frac{\partial\sqrt{G}}{\partial r}.$$

This with $\sqrt{G}(0,\theta)=0$, $\frac{\partial\sqrt{G}}{\partial r}(0,\theta)=1$ yields

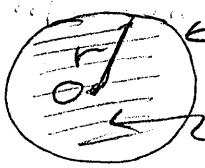
$$\sqrt{G(r,\theta)} = r - \frac{K_0}{6}r^3 + O(r^4)$$

and

$$G(r,\theta) = r^2 - \frac{K_0}{3}r^4 + O(r^5).$$

Geodesic circles

- circumference & area



$$L(r) = \int_0^{2\pi}\sqrt{G(r,\theta)}\,d\theta$$

$$= \int_0^{2\pi}\left(r - \frac{K_0}{6}r^3 + O(r^4)\right)d\theta$$

$$= 2\pi\left(r - \frac{K_0}{6}r^3\right) + O(r^4),$$

so $$\frac{2\pi r - L(r)}{\frac{\pi r^3}{6}} = K_0 + O(r).$$

and $$\boxed{K_0 = \lim_{r \to 0} \frac{2\pi r - L(r)}{\frac{\pi r^3}{3}}}$$ Bertrand & Puiseux 1848

And,

$$A(r) = \iint_{\substack{0 \le \rho \le r \\ 0 \le \theta \le 2\pi}} \sqrt{G(\rho,\theta)}\, d\rho\, d\theta \qquad \leftarrow EG - F^2 = G$$

$$= \iint \left(\rho - \frac{K_0}{6}\rho^3 + O(\rho^4)\right) d\rho\, d\theta$$

$$= 2\pi\left(\frac{1}{2}r^2 - \frac{K_0}{24}r^4\right) + O(r^5)$$

so

$$\frac{\pi r^2 - A(r)}{\frac{\pi r^4}{12}} = K_0 + O(r)$$

and

$$\boxed{K_0 = \lim_{r \to 0} \frac{\pi r^2 - A(r)}{\frac{\pi r^4}{12}}}.$$

Remark These two formulas show how to find curvature intrinsically - geometrically.

Also, as $K_0 \gtrless 0$ or < 0, circumference and area either fall short of or exceed their Euclidean values.

These formulas shows - by "how much".

Surfaces of constant curvature - local

Theorem Any two surfaces having the same constant Gaussian curvature are isometric in the small.

Proof Case that $K = -\frac{1}{h^2}$ is negative:
The formula $K = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial r^2}$ gives
the initial value problem (for any θ fixed):

$$\begin{cases} \frac{\partial^2}{\partial r^2}\sqrt{G} + \frac{1}{h^2}\sqrt{G} = 0 \\ \sqrt{G} = 0 \\ \frac{\partial \sqrt{G}}{\partial r} = 1 \end{cases} \quad \text{at } r=0$$

The solution is $\sqrt{G} = h \sinh \frac{r}{h}$
The positive and zero curvature cases
are analogous. q.e.d.

Problem (7.4) Verify - as in the theorem -
that one obtains the three "model"
local metrics

$$ds^2 = dr^2 + \left(a \sin \tfrac{r}{a}\right)^2 d\theta^2 \quad , \; K = \tfrac{1}{a^2}$$

$$ds^2 = dr^2 + r^2 d\theta^2 \quad , \; K = 0$$

$$ds^2 = dr^2 + \left(h \sinh \tfrac{r}{h}\right)^2 d\theta^2 \quad , \; K = -\tfrac{1}{h^2}.$$

ADDITION TO LECTURE 7

Properties of the intrinsic derivative.

$C(\tau)$, $a \le \tau \le b$ curve on an abstract surface M

$V(\tau)$, $W(\tau)$ vector fields along C.

1) $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$

$\frac{D}{dt}(aV) = a\frac{DV}{dt}$ a = const.

Proof is trivial.

2) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$, $f = f(\tau)$ differentiable function along C.

Proof In a patch: C is $u^i = u^i(\tau)$;

$$\left(\frac{D}{dt}(fV)\right)^i = \frac{d}{dt}(fV^i) + \sum \Gamma^i_{rs}(fV^r)\frac{du^s}{dt}$$

$$= \frac{df}{dt}V^i + f\left(\frac{dV^i}{dt} + \sum \Gamma^i_{rs}V^r\frac{du^s}{dt}\right)$$

$V \circ W$

$$= \left(\frac{df}{dt}V + f\frac{DV}{dt}\right)^i \quad , i = 1,2.$$

3). $\frac{d}{dt}\underbrace{\langle V, W\rangle}_{\text{function along } C} = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle$

Proof.

$$\langle V, W\rangle = \sum g_{ij}V^iW^j.$$

Will need $\frac{\partial g_{ij}}{\partial u^k}$!

$$\Gamma^i_{jk} = \frac{1}{2}g^{ir}\left(\frac{\partial g_{kr}}{\partial u^j} + \frac{\partial g_{jr}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^r}\right)$$ (summation understood)

$$\frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i} = 2 g_{ir} \Gamma^r_{jk}$$

$$\frac{\partial g_{ji}}{\partial u^k} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ik}}{\partial u^j} = 2 g_{jr} \Gamma^r_{ik} \qquad (i \text{ and } j \text{ interchanged})$$

Add and divide by 2:

$$\frac{\partial g_{ij}}{\partial u^k} = g_{ir} \Gamma^r_{jk} + g_{jr} \Gamma^r_{ik} .$$

$$\frac{d}{dt} \langle V, W \rangle = \frac{d}{dt} (g_{ij} V^i W^j) \qquad (\text{summation understood})$$

$$= \frac{\partial g_{ij}}{\partial u^k} \frac{du^k}{dt} V^i W^j + g_{ij} \frac{dV^i}{dt} W^j + g_{ij} V^i \frac{dW^j}{dt}$$

$$= g_{ir} \Gamma^r_{jk} \frac{du^k}{dt} V^i W^j + g_{jr} \Gamma^r_{ik} \frac{du^k}{dt} V^i W^j$$

$$+ \text{—}''\text{—} + \text{—}''\text{—}$$

$$= g_{ij} \left(\frac{dV^i}{dt} + \Gamma^i_{jk} \right)$$

$$= g_{jr} \left(\frac{dV^r}{dt} + \Gamma^r_{ik} V^i \frac{du^k}{dt} \right) W^j + g_{ir} V^i \left(\frac{dW^r}{dt} \right)$$

$$= g_{jr} \left(\frac{dV^r}{dt} + \Gamma^r_{ik} V^i \frac{du^k}{dt} \right) W^j$$

$$+ g_{ir} V^i \left(\frac{dW^r}{dt} + \Gamma^r_{jk} W^j \frac{du^k}{dt} \right)$$

$$= \left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{dW}{dt} \right\rangle .$$

Next: For surfaces in space this can be verified using $\frac{dV}{dt} \cdot W = \frac{DV}{dt} \cdot W$ etc., as W is tangent