Example. Surfaces of revolution

See Lecture 4 pp. 4-18 ff.

Profile curve. u is **not** necessarily the arc length.

Denote $\frac{d}{du}$ by $'$, $''$.

$$S(u,\theta) = \begin{bmatrix} g(u)\cos\theta \\ g(u)\sin\theta \\ h(u) \end{bmatrix}$$

$$\begin{cases} u^1 = u \\ u^2 = \theta \end{cases}$$

Metric and normal

$$S_1 = \begin{bmatrix} \dot g\cos\theta \\ \dot g\sin\theta \\ \dot h \end{bmatrix}, \quad S_2 = \begin{bmatrix} -g\sin\theta \\ g\cos\theta \\ 0 \end{bmatrix}, \quad \begin{aligned} E &= \dot g^2 + \dot h^2 \\ F &= 0 \\ G &= g^2 \end{aligned}$$

$$S_1 \times S_2 = \begin{bmatrix} -g\dot h\cos\theta \\ g\dot h\sin\theta \\ g\dot g \end{bmatrix}, \quad n = \begin{bmatrix} -\dot h\cos\theta \\ -\dot h\sin\theta \\ \dot g \end{bmatrix} \frac{1}{\sqrt{\dot g^2+\dot h^2}}$$

$$\|S_1 \times S_2\| = g\sqrt{\dot g^2+\dot h^2}$$

Christoffel symbols

$$S_{11} = \begin{bmatrix} \ddot g\cos\theta \\ \ddot g\sin\theta \\ \ddot h \end{bmatrix}, \quad S_{12} = \begin{bmatrix} -\dot g\sin\theta \\ \dot g\cos\theta \\ 0 \end{bmatrix}, \quad S_{22} = \begin{bmatrix} -g\cos\theta \\ -g\sin\theta \\ 0 \end{bmatrix}$$

From

$$S_{ij} = S_a \Gamma_{ij}^a + n L_{ij} \quad \text{and} \quad g_{ra} = S_r \cdot S_a$$

obtain

$$g_{ra}\Gamma_{ij}^a = S_r \cdot S_{ij}.$$

So

$$\begin{bmatrix} \dot g^2 + \dot h^2 & 0 \\ 0 & g^2 \end{bmatrix} \begin{bmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \end{bmatrix} = \begin{bmatrix} S_1 \cdot S_{ij} \\ S_2 \cdot S_{ij} \end{bmatrix}.$$

$S_r \cdot S_{ij}$	$S_1 S_{11}$	$S_{12} = S_{21}$	S_{22}
S_1	$\dot g\ddot g + \dot h\ddot h$	0	$-g\dot g$
S_2	0	$g\dot g$	0

So

$$\Gamma^1_{11} = \frac{\dot g \ddot g + \dot u \ddot u}{\dot g^2 + \dot u^2}, \qquad \Gamma^1_{22} = -\frac{g\dot g}{\dot g^2 + \dot u^2},$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{\dot g}{g}, \quad \text{all others are } \underline{\text{zero}}.$$

Geodesic equations

$$\frac{d^2 u^a}{ds^2} + \Gamma^a_{ib}(u^1, u^2)\frac{du^i}{ds}\frac{du^b}{ds} = 0 \qquad \text{summed on } i \text{ & } b$$

$u^1 = u, \quad u^2 = \theta$

Obtain

$$\begin{cases} \dfrac{d^2u}{ds^2} + \dfrac{\dot g \ddot g + \dot u \ddot u}{\dot g^2 + \dot u^2}\left(\dfrac{du}{ds}\right)^2 - \dfrac{g\dot g}{\dot g^2 + \dot u^2}\left(\dfrac{d\theta}{ds}\right)^2 = 0 \\ \dfrac{d^2\theta}{ds^2} + 2\dfrac{\dot g}{g}\dfrac{du}{ds}\dfrac{d\theta}{ds} = 0 \end{cases}$$

$\Gamma^1_{12} = \Gamma^1_{21}$

First integral or "conserved quantity".

Since s is arc length on a geodesic, we have

$$(\dot g^2 + \dot u^2)\left(\frac{du}{ds}\right)^2 + g^2\left(\frac{d\theta}{ds}\right)^2 = 1.$$

Quadrature Of the two equations for a geodesic one, along with the first integral, suffices. Take the second.

$$g^2\frac{d^2\theta}{ds^2} + \underbrace{2g\dot g\frac{du}{ds}}_{\frac{d}{ds}(g(u)^2)}\frac{d\theta}{ds} = 0, \quad \text{uses } g \neq 0$$

so

$$\frac{d}{ds}\left(g^2\frac{d\theta}{ds}\right) = 0 \quad \text{or} \quad g^2\frac{d\theta}{ds} = c, \text{ CONSTANT.}$$

Divide first integral by this last equation squared; and use $\frac{du}{ds}\Big/\frac{d\theta}{ds} = \frac{du}{d\theta}$ on a solution curve!

$$\frac{\dot g^2 + \dot u^2}{g^4}\left(\frac{du}{d\theta}\right)^2 + \frac{1}{g^2} = \frac{1}{c^2}.$$

From this obtain

$$\pm d\theta = \frac{c\sqrt{g'^2 + h'^2}}{g\sqrt{g^2 - c^2}}\, du$$

so

$$\pm\theta + \text{CONST} = \int \frac{c}{g}\,\frac{\sqrt{g'^2 + h'^2}}{\sqrt{g^2 - c^2}}\, du$$

The integrand is a function of u only. Thus: The geodesics on a surface of revolution can be found by quadrature.

Problem (6.12) A sphere of radius a has profile curve $g(u) = a\sin u$, $h(u) = a\cos u$, u = colatitude = $90° -$ latitude.

a) Show that the differential equations for geodesics is the coupled system

$$\begin{cases} \dfrac{d^2u}{ds^2} + \sin u \cos u \left(\dfrac{d\theta}{ds}\right)^2 = 0 \\ \dfrac{d^2\theta}{ds^2} + 2\dfrac{\cos u}{\sin u}\dfrac{du}{ds}\dfrac{d\theta}{ds} = 0 \end{cases}$$

b) Solve the system in a). Verify that geodesics are great circles lying in planes $A\,a\sin u(s) + B\,a$

$$A\,a\sin u(s)\cos\theta(s) + B\,a\sin u(s)\sin\theta(s) + C\,a\cos u(s) = 0.$$

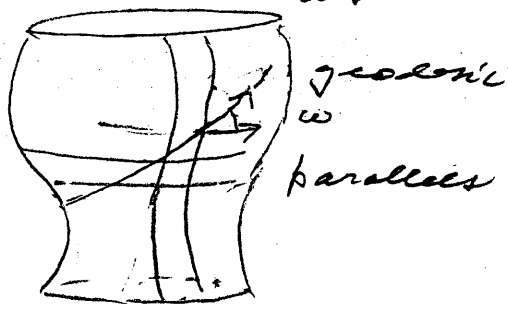
Problem (6.13) Find the geodesics on a right circular cone.

Problem (6.14) Show that the geodesics on the right helicoid

$$S(u,v) = \begin{vmatrix} u\cos v \\ u\sin v \\ av \end{vmatrix}$$

can be found by means of elliptic integrals. Draw appropriate figures.

Theorem of Clairaut (1733)



$$S(u,\theta) = \begin{bmatrix} g(u)\cos\theta \\ g(u)\sin\theta \\ h(u) \end{bmatrix}$$

Parallel curve: $S(\text{CONST}, \theta)$.

Tangent to parallel:

$$\frac{d}{d\theta} S(\text{CONST}, \theta) = S_2 = \begin{bmatrix} -g\sin\theta \\ g\cos\theta \\ 0 \end{bmatrix}$$

Arbitrary curve:

$$S(u(s), \theta(s)) \qquad \begin{cases} u = u(s) \\ \theta = \theta(s) \end{cases}$$

s = arc length

Tangent to this curve is the unit vector

$$\frac{d}{ds} S(u(s), \theta(s)) = S_1 \frac{du}{ds} + S_2 \frac{d\theta}{ds}.$$

Angle between this curve and parallel at $S(u(s), \theta(s))$ is $\omega(s)$:

$$\cos\omega(s) = \frac{S_2 \cdot \left(S_1 \frac{du}{ds} + S_2 \frac{d\theta}{ds}\right)}{\|S_2\| \, \| \text{---} u \text{---} \|}$$

$$= \frac{0 \cdot \frac{du}{ds} + g^2 \frac{d\theta}{ds}}{g \cdot 1} = g \frac{d\theta}{ds}.$$

If the curve is a geodesic, $g^2 \frac{d\theta}{ds} = \text{CONSTANT}$, so

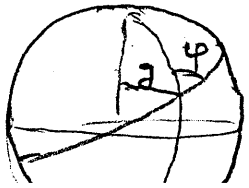
$$g(u(s)) \cos\omega(s) = \text{CONSTANT}$$

or

$$\begin{pmatrix} \text{distance} \\ \text{to central} \\ \text{axis} \end{pmatrix} \cdot \sin\left(\begin{matrix} \text{angle between} \\ \text{curve \& meridians} \end{matrix}\right)$$

is a constant along a geodesic.

Eg Sphere



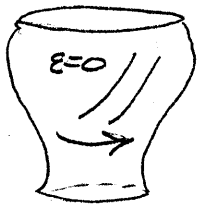
g smaller for φ larger.

Remark Clairaut's theorem can also be obtained by a variation. Suppose $\sigma(t,\varepsilon)$ is a variation with all of the curves $\sigma(\cdot,\varepsilon)$ being geodesics. Then

$$\frac{dL}{d\varepsilon}(0) = \left[\frac{\partial\sigma}{\partial\varepsilon}(t,0)\cdot\frac{\partial\sigma}{\partial s}(t,0)\right]_{t=0}^{t=l} - \int_0^l \frac{\partial\sigma}{\partial\varepsilon}(t,0)\cdot\frac{\partial^2\sigma}{\partial t^2}(t,0)\,dt,$$

and $$\frac{\partial\sigma}{\partial\varepsilon}(t,0)\cdot\frac{\partial^2\sigma}{\partial t^2}(t,0) = 0$$

since $\frac{\partial\sigma}{\partial\varepsilon}(t,0)$ is tangent to the surface S and $\frac{\partial^2\sigma}{\partial t^2}(t,0)$ is normal to S.



A rotation is a variation through geodesics and the length $L(\varepsilon)$ does not change, so $\left[\frac{\partial\sigma}{\partial\varepsilon}(t,0)\cdot\frac{\partial\sigma}{\partial t}(t,0)\right]_{t=0}^{t=l} = 0$

Hence $$\frac{\partial\sigma}{\partial\varepsilon}(t,0)\cdot\frac{\partial\sigma}{\partial t}(t,0) = \text{CONSTANT}$$

The rotation is

$$\sigma(t,\varepsilon) = S(u(t),\theta(t)+\varepsilon) = \begin{bmatrix} g(u(t))\cos(\theta(t)+\varepsilon) \\ g(u(t))\sin(\theta(t)+\varepsilon) \\ h(u(t)) \end{bmatrix}$$

This gives

$$\frac{\partial\sigma}{\partial\varepsilon}(t,0) = S_2(u(t),\theta(t)) \qquad \frac{\partial\sigma}{\partial t}(t,0) = S_1(u,\theta)u' + S_2(u,\theta)\theta'$$

with product

$$S_2(u(t),\theta(t))\cdot(S_1(u,\theta)u' + S_2(u,\theta)\theta') = \text{const}$$

$$S_2\cdot S_2 = g^2$$

so $$g(u(t))^2\frac{d\theta}{ds}(t) = \text{CONSTANT}$$

Finally $\cos\omega(t) = \dfrac{S_2\cdot(S_1u' + S_2\theta')}{\|S_2\|\,\|\text{---}\,u\,\text{---}\|}$

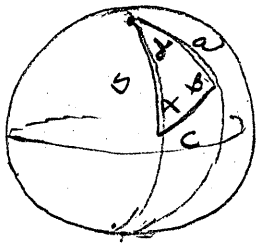
as before.

Problem (6.15) Prove: On a surface of revolution, if ω is the angle between (the tangent to) a curve and the circle of latitude at a point and that circle has radius g, then $g \cos\omega = \text{CONSTANT}$ implies that the curve is a geodesic. (Converse of Clairaut's Theorem.)

Problem (6.16) On a sphere, take a spherical triangle with one vertex at the "north pole" and the other two vertices on meridians from the north to the south poles. Use Clairaut's Theorem to show that

$$\frac{\sin a}{\sin\alpha} = \frac{\sin b}{\sin\beta} = \frac{\sin c}{\sin\gamma},$$

the Law of Sines for spherical triangles.



Problem (6.17) On a surface of revolution, a curve making a constant angle with the meridian lines is called a loxodrome.

a) Find a differential equation for loxodromes. Show that its solutions may be obtained by quadrature.

b) Show that loxodromes on the sphere

$$S(r,\theta) = \begin{vmatrix} a\sin\frac{r}{a}\cos\theta \\ a\sin\frac{r}{a}\sin\theta \\ a\cos\frac{r}{a} \end{vmatrix}$$

$\frac{r}{a}$ = co-latitude
θ = longitude
$0 < r < \pi a$

are given by

$$\pm(\theta + \text{CONST})\cot\alpha = \ln\tan\frac{r}{2a}$$

These curves "spiral" around the poles. Make a careful picture. Such curves are also called rhumb lines.