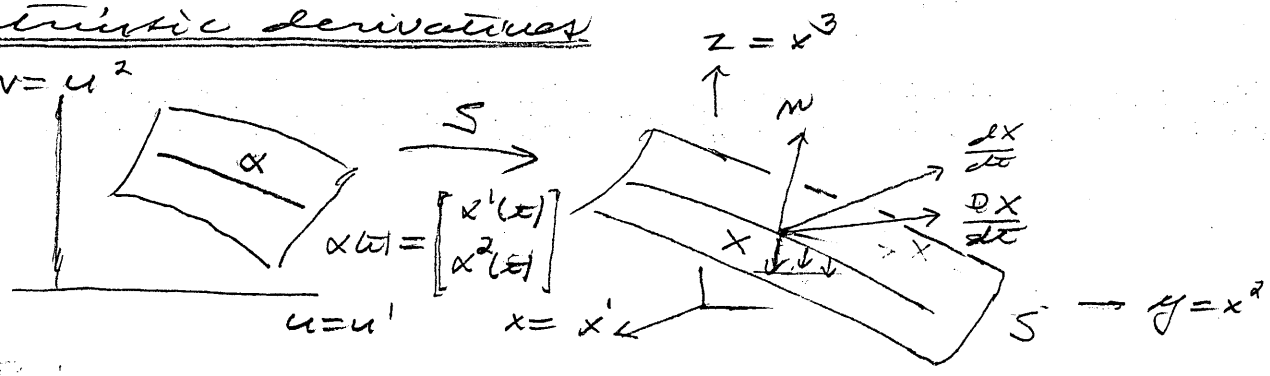
3/2/01 **Intrinsic derivatives**



Curve: The intrinsic derivative is of vector fields along a given curve on S:

$S \circ \alpha$ or $S(\alpha^1(t), \alpha^2(t))$

Note: $x \mapsto S(\alpha(t))$ is locally injective, so function on the curve can be written as a function of the parameter t.

Vector field — along $S \circ \alpha$ which is tangent to S:

$$X(t) = S_1(\alpha(t))\, X^1(t) + S_2(\alpha(t))\, X^2(t)$$

$\frac{\partial S}{\partial u}(\alpha(t))$ — function of t

Derivatives

$$\frac{dX}{dt}(t) = \left(\begin{array}{l}\text{ordinary "component wise"} \\ \text{derivative of } X(t) \text{ in Euclidean } \mathbb{E}^3\end{array}\right)$$

Note: This vector is not necessarily tangent to the surface.

$$\frac{DX}{dt}(t) = \left(\begin{array}{l}\text{projection of } \frac{dX}{dt}(t) \\ \text{to the tangent plane} \\ \text{at } S(\alpha(t))\end{array}\right) \quad \begin{array}{l}\text{vector} \\ \text{field} \\ \text{on } S\circ\alpha\end{array}$$

We have

$$\frac{DX}{dt} = \frac{dX}{dt} - N\left(N \cdot \frac{dX}{dt}\right), \quad N = N(\alpha(t))$$

Definition: $\frac{DX}{dt}$ is the intrinsic or covariant derivative of X along $S\circ\alpha$.

Later: $\frac{DX}{dt}$ depends only on the metric E, F, G and does not need $\frac{dX}{dt}$.

Calculus $\frac{D}{dt}$ along $S\circ\alpha$:

1) is linear. For a & b constants,

$$\frac{D}{dt}(X(t)a + Y(t)b) = \frac{DX}{dt}(t)a + \frac{DY}{dt}(t)b.$$

2) satisfies Leibniz rule. For f a function on $S\circ\alpha$,

$$\frac{D}{dt}(X(t)f(t)) = \frac{DX}{dt}(t)f(t) + X(t)\frac{df}{dt}(t).$$

For: $\frac{d}{dt}(Xf) = \frac{dX}{dt}f + X\frac{df}{dt}$.

Now subtract $N(M\cdot(-1))$ noting $N\cdot X = 0$.

3) Respects the inner product:

$$\frac{d}{dt}(X(t)\cdot Y(t)) = \frac{DX}{dt}(t)\cdot Y(t) + X(t)\cdot\frac{DY}{dt}(t)$$

Note: All vectors $X, Y, \frac{DX}{dt}, \frac{DY}{dt}$ are tangent to S and "$\cdot$" needs only E, F, G.

Problem (6-5) Prove 3) above.

Geodesic curvature

$S(\alpha(s))$ curve on surface S

s = arc length as parameter

$T = \frac{d}{ds}S(\alpha(s))$ unit tangent

$\frac{dT}{ds} = U\kappa_g + N\kappa_N$ curvature vector

Take tangential component:

$$\frac{DT}{ds} = U\kappa_g \quad \text{or} \quad \kappa_g = \frac{DT}{ds}\cdot U$$

Frenet equations Assumptions as above.

$U\cdot U = 1$ so $2U\cdot\frac{DU}{dt} = 0$

$T\cdot U = 0$ so $\underbrace{\frac{DT}{ds}}_{\kappa_g}\cdot U + T\cdot\frac{DU}{ds} = 0$

so $\frac{DU}{ds} = -T\kappa_g$.

Frenet equations:
$$\begin{cases} \frac{DT}{ds} = U\kappa_g \\ \frac{DU}{ds} = -T\kappa_g \end{cases}$$

Geodesics σ is a geodesic on S exactly when $\frac{D}{ds}\frac{d\sigma}{ds} = 0$. This will be seen to be intrinsic.
s = arc length

Indices
- Einstein summation convention: Repeated indices, one up & one down, are understood as being summed over 1 and 2.
  Eg $(BA)^i{}_j = B^i{}_r A^r{}_j$ for matrix product.
- Raising & lowering indices.

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \quad \text{metric at a point}$$

$$\begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{G}{EG-F^2} & -\frac{F}{EG-F^2} \\ -\frac{F}{EG-F^2} & \frac{E}{EG-F^2} \end{bmatrix}$$

so $g^{ir} g_{rj} = \delta^i{}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$ Kronecker delta

Eg X^i vector $\begin{bmatrix} X^1 \\ X^2 \end{bmatrix}$, $X_i = g_{ir} X^r$ covector $[X_1 \; X_2]$

Problem (6.6) For vectors X and Y, check that
$X \cdot Y = g_{ij} X^i Y^j = X_a Y^a = X^b Y_b = g^{rs} X_r Y_s$.
for $EX^1X^2 + F(X^1Y^2 + X^2Y^1) + GX^2Y^2$.

Gauss equations

$S(u^1, u^2)$ regular surface $u^1 = u$, $u^2 = v$

$$S_i(u^1, u^2) = \frac{\partial S}{\partial u^i}(u^1, u^2) \qquad i = 1, 2$$

At a point $S(u^1, u^2)$:

$\underbrace{S_1(u^1, u^2), S_2(u^1, u^2)}$, $n(u^1, u^2)$ basis of E^3

basis of tangent plane; unit normal

is a basis of Euclidean vector space E^3.

Write

$$\frac{\partial S_b}{\partial u^i}(u^1,u^2) = \sum_{a=1}^{2} S_a(u^1,u^2)\,\underbrace{\Gamma_{ib}^{a}(u^1,u^2)}_{\text{Christoffel symbols}} + N(u^1,u^2)\,\underbrace{L_{ib}(u^1,u^2)}_{\text{second fundamental form } L,\Pi,N}$$

or

$$S_{ib} = S_a \Gamma_{ib}^{a} + N L_{ib} \qquad \begin{matrix} i=1,2 \\ b=1,2 \end{matrix}.$$

These are the Gauss equations which define Γ_{ib}^{a} (and L_{ib}).

Remark Vectors X^i, forms g_{ij} and L_{ib} are "tensors". The Christoffel symbols Γ_{ib}^{a} are not tensors - they "transform" differently under change of coordinates.

Coordinates for the intrinsic derivative

$\begin{cases} u^1 = \alpha^1(t) \\ u^2 = \alpha^2(t) \end{cases}$ curve $S(\alpha(t))$ on surface S.

Velocity vector

$$\frac{d}{dt} S(\alpha(t)) = S_1(\alpha^1(t),\alpha^2(t))\,\dot\alpha^1(t) + S_2(-,-)\,\dot\alpha^2(t)$$
$$= S_i\,\dot\alpha^i \quad \text{summed on } i=1,2$$

Vector field along curve:

$$X(t) = S_1(\alpha^1(t),\alpha^2(t))\,X^1(t) + S_2(-,-)\,X^2(t)$$
$$= S_a X^a \quad \text{summed on } a=1,2$$

Take derivative of $X(t)$ in E^3:

$$\frac{dX}{dt} = \frac{d}{dt}(S_a X^a) = \left(\frac{d}{dt} S_b\right) X^b + S_a \frac{dX^a}{dt}$$
$$= (S_{ib}\dot\alpha^i) X^b + S_a \dot X^a$$
$$= (S_a \Gamma_{ib}^{a} + N L_{ib})\,\dot\alpha^i X^b + S_a \dot X^a$$
$$= \underbrace{S_a\left(\dot X^a + \Gamma_{ib}^{a}\dot\alpha^i X^b\right)}_{\text{tangent to surface}} + \underbrace{N L_{ib}\dot\alpha^i X^b}_{\text{normal to surface}}$$

The component of $\frac{dX}{dt}$ tangent to the surface is:

$$\frac{DX}{dt} = S_a \left(\frac{dX^a}{dt} + \Gamma^a_{ib} \frac{du^i}{dt} \frac{dX^b}{dt} \right)$$ summed on i, a, b over $1, 2$

or, the a^{th} component is

$$\boxed{\left(\frac{DX}{dt}\right)^a = \dot{X}^a + \Gamma^a_{ib} \dot{u}^i X^b}$$

Geodesics

$\begin{cases} u^1 = \alpha^1(s) \\ u^2 = \alpha^2(s) \end{cases}$ geodesic $S(\alpha(s))$ on S, s = arc length

$$T(s) = \frac{d}{ds} S(\alpha(s)) = S_a(\alpha(s)) \alpha'^a(s), \quad ' \text{ is } \frac{d}{ds}$$

- unit tangent along $S \circ \alpha$

$$\frac{DT}{ds} = S_a \left(\alpha''^a + \Gamma^a_{ib} \alpha'^i \alpha'^b \right)$$

Thus $\frac{DT}{ds} = 0$ gives the system

$$\boxed{\frac{d^2 u^a}{ds^2} + \sum_{i,b=1}^{2} \Gamma^a_{ib}(u^1, u^2) \frac{du^i}{ds} \frac{du^b}{ds} = 0 \qquad a = 1, 2}$$

of two coupled non-linear second order ordinary differential equations whose solution is a geodesic $u^1 = \alpha^1(s)$, $u^2 = \alpha^2(s)$ parameterized by arc length.

Problem (6.7) Find equations for geodesics $S(\alpha(u))$ having parameter u which is not necessarily arc length.

Hint: $\frac{DX}{dt} = \frac{DX}{ds} \frac{ds}{dt}$.

Computation of Christoffel symbols

$$S_{ij} = S_\alpha \Gamma_{ij}^{\alpha} + n L_{ij}$$

Take inner product with vector $S_r = \frac{\partial S}{\partial u^r}$:

$$S_r \cdot S_{ij} = \underbrace{S_r \cdot S_\alpha}_{g_{r\alpha}} \Gamma_{ij}^{\alpha} + \underbrace{S_r \cdot n}_{0} L_{ij}.$$

So

$$g_{r\alpha} \Gamma_{ij}^{\alpha} = S_r \cdot S_{ij} \quad \text{or} \quad \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \end{bmatrix} = \begin{bmatrix} S_1 \cdot S_{ij} \\ S_2 \cdot S_{ij} \end{bmatrix}$$

and

$$\Gamma_{ij}^{\alpha} = g^{\alpha r} S_r \cdot S_{ij} \quad \text{or} \quad \begin{bmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{-1} \begin{bmatrix} S_1 \cdot S_{ij} \\ S_2 \cdot S_{ij} \end{bmatrix}.$$

Christoffel symbols – intrinsic

Begin with $g_{ir} = S_i \cdot S_r$ and take the partial derivatives:

$$(+) \quad \frac{\partial g_{ir}}{\partial u^j} = \frac{\partial}{\partial u^j}(S_i \cdot S_r) = S_{ji} \cdot S_r + S_i \cdot S_{jr}$$

$$(-) \quad \frac{\partial g_{ij}}{\partial u^r} = S_{ri} \cdot S_j + S_i \cdot S_{rj}$$

$$(+) \quad \frac{\partial g_{rj}}{\partial u^i} = S_{ir} \cdot S_j + S_r \cdot S_{ij}$$

Add and subtract as indicated:

$$\frac{\partial g_{ir}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^r} + \frac{\partial g_{rj}}{\partial u^i} = 2 S_r \cdot S_{ij}$$

Using $\Gamma_{ij}^{\alpha} = g^{\alpha r} S_r \cdot S_{ij}$, we have

$$\Gamma_{ij}^{\alpha} = \frac{1}{2} g^{\alpha r} \left(\frac{\partial g_{ir}}{\partial u^j} + \frac{\partial g_{jr}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^r} \right) \quad \text{summed on } r$$

for $i, j = 1, 2$.

Note this says the Christoffel symbols are intrinsic – they depend on the first fundamental form only.

Since $\Gamma^a_{ij} = \Gamma^a_{ji}$, there are just six different symbols. In terms of E, F, G, $E_1 = \frac{\partial E}{\partial u^1}$, etc, and $W^2 = EG - F^2$, they are

$$\Gamma^1_{11} = \frac{GE_1 - 2FF_1 + FE_2}{2W^2} \qquad \Gamma^2_{11} = \frac{-FE_1 + 2EF_1 - EE_2}{2W^2}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{GE_2 - FG_1}{2W^2} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{-FE_2 + EG_1}{2W^2}$$

$$\Gamma^1_{22} = \frac{2GF_2 - GG_1 - FG_2}{2W^2} \qquad \Gamma^2_{22} = \frac{-2FF_2 + FG_1 + EG_2}{2W^2}$$

Consequence The intrinsic derivative

$$\left(\frac{DX}{dt}\right)^a = \frac{dX^a}{dt} + \Gamma^a_{ij}\frac{du^i}{dt}X^j$$

is intrinsic - it depends only on the first fundamental form.

Problem (6.8) check the formulas above for the Christoffel symbols.

Example The plane in polar coördinates.

$ds^2 = dr^2 + r^2 d\theta^2$ $\quad E = 1,\ F = 0,\ G = r^2.$

One has: $u^1 = u = r$, $u^2 = v = \theta$

$$\Gamma^1_{11} = 0 \qquad \Gamma^2_{11} = 0$$

$$\Gamma^1_{12} = \Gamma^1_{21} = 0 \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$$

$$\Gamma^1_{22} = -r \qquad \Gamma^2_{22} = 0$$

The equations for a geodesic are

$$\begin{cases} \dfrac{d^2r}{ds^2} - r\left(\dfrac{d\theta}{ds}\right)^2 = 0 \\ \dfrac{d^2\theta}{ds^2} + 2\dfrac{1}{r}\dfrac{dr}{ds}\dfrac{d\theta}{ds} = 0 \end{cases}$$

(Γ^1_{12} and Γ^2_{21})

First integral or "conserved quantity":

$ds^2 = dr^2 + r^2 d\theta^2$ or $\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = 1$.

Another first integral: second equation times r^2 is

$$r^2 \frac{d^2\theta}{ds^2} + 2r \frac{dr}{ds}\frac{d\theta}{ds} = 0 \quad \text{or} \quad \frac{d}{ds}\left(r^2 \frac{d\theta}{ds}\right) = 0$$

so $r^2 \frac{d\theta}{ds} = c$, CONSTANT.

Divide $\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2 = 1$

by $\left(r^2 \frac{d\theta}{ds}\right)^2 = c^2$

to obtain $\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2} = \frac{1}{c^2}$, using $\frac{\frac{dr}{ds}}{\frac{d\theta}{ds}} = \frac{dr}{d\theta}$.

From this

$$\pm d\theta = \frac{dr}{r\sqrt{\frac{r^2}{c^2} - 1}}$$

Integrate:

$$\pm\theta + \text{CONST} = \pm \operatorname{Arcsec} \frac{r}{c}.$$

The equation

$$r = c \sec(\pm\theta + \text{CONST}),$$

is a straight line in polar coördinates.

Problem (6.9) a) Check that the first equation $\frac{d^2r}{ds^2} - r\left(\frac{d\theta}{ds}\right)^2 = 0$ is also satisfied.

b) Obtain solutions using the first equation and $ds^2 = dr^2 + r^2 d\theta^2$.

Problem (6.10) Show that the geodesics on a cylinder are helices.

Problem (6.11) Find the equations for geodesics on a surface in range form $z = f(x, y)$.