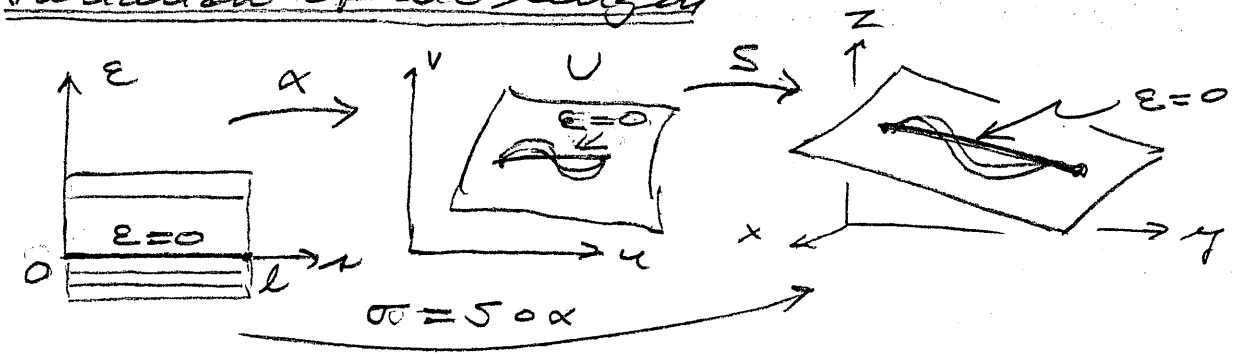
1/26/01 **6. Surfaces in space. Geodesics**

Variation of arc length



One parameter family of curves

$$\sigma(t,\varepsilon) = (S\circ\alpha)(t,\varepsilon) = S(\alpha^1(t,\varepsilon), \alpha^2(t,\varepsilon))$$

Assumptions:

- all functions are smooth
- $\sigma(-,0)$ curve being varied
  - typically to minimize arc length
- $\sigma(-,\varepsilon)$ curves making up variation
- s arc length along $\sigma(-,0)$
  - not necessarily arc length along $\sigma(-,\varepsilon)$ for $\varepsilon \neq 0$.

And, in case one or both endpoints are fixed:

- $\sigma(0,\varepsilon)$ = CONSTANT & $\sigma(l,\varepsilon)$ = CONSTANT

where l = length of $\sigma(-,0)$.

Arc length of curve $\sigma(-,\varepsilon)$:

$$L(\varepsilon) = \int_0^l \sqrt{\frac{\partial\sigma}{\partial s}(t,\varepsilon)\cdot\frac{\partial\sigma}{\partial s}(t,\varepsilon)}\, ds$$

Then

$$\frac{dL}{d\varepsilon}(\varepsilon) = \int_0^l \frac{1}{2\underbrace{\sqrt{\frac{\partial\sigma}{\partial s}\cdot\frac{\partial\sigma}{\partial s}}}_{=1 \text{ at } \varepsilon=0}} \underbrace{\frac{\partial}{\partial\varepsilon}\left(\frac{\partial\sigma}{\partial s}\cdot\frac{\partial\sigma}{\partial s}\right)}_{2\left(\frac{\partial}{\partial\varepsilon}\frac{\partial\sigma}{\partial s}\right)\cdot\frac{\partial\sigma}{\partial s}} ds$$

$$\frac{\partial}{\partial s}\frac{\partial\sigma}{\partial\varepsilon}$$

so

$$\frac{dL}{d\varepsilon}(0) = \int_0^l \left(\frac{\partial}{\partial s}\frac{\partial\sigma}{\partial\varepsilon}(t,0)\right)\cdot\frac{\partial\sigma}{\partial s}(t,0)\, ds$$

Integrate by parts:

$$\frac{dL}{d\varepsilon}(0) = \left[\frac{\partial \sigma}{\partial \varepsilon}(s,0) \cdot \frac{\partial \sigma}{\partial s}(s,0) \right]_{s=0}^{s=l} - \int_0^l \frac{\partial \sigma}{\partial \varepsilon}(s,0) \cdot \frac{\partial^2 \sigma}{\partial s^2}(s,0)\, ds$$

Nomenclature

$\frac{dL}{d\varepsilon}(0)$ — first variation of length of the curve $\sigma(\cdot,0)$.

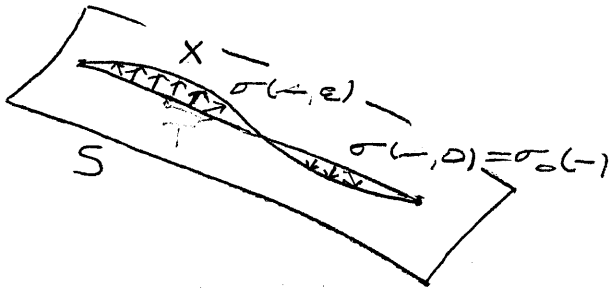
$\frac{\partial \sigma}{\partial s}(s,0) = T(s)$ — unit tangent along $\sigma(\cdot,0)$ - tangent to surface

$\frac{\partial^2 \sigma}{\partial s^2}(s,0) = \frac{dT}{ds}(s)$ — curvature vector κ_N along $\sigma(\cdot,0)$, acceleration - not necessarily tangent to surface

$\frac{\partial \sigma}{\partial \varepsilon}(s,0) = J(s)$ — variation vector along $\sigma(\cdot,0)$ - tangent to surface

Fundamental lemma

Let $\sigma_0(s)$ be a curve, on the surface S, parameterized by arc length s, $0 \le s \le l$. Let $X(s)$ be a vector field along σ_0 which is tangent to the surface S and which vanishes at the endpoints $s=0$ and $s=l$. Then: there exists a variation $\sigma(s,\varepsilon)$ such that:



$\sigma(s,0) = \sigma_0(s)$ — varies σ_0

$\sigma(0,\varepsilon) = \text{CONSTANT}$
$\sigma(l,\varepsilon) = \text{CONSTANT}$ } — end-points fixed

and

$\frac{\partial \sigma}{\partial \varepsilon}(s,0) = X(s)$, — variation vector is X

for $0 \le s \le l$

Proof We have $\sigma_0(t) = S(\alpha^1(t), \alpha^2(t))$ and

$$X(t) = S_1(\alpha^1(t), \alpha^2(t))\, X^1(t) + S_2(\alpha^1(t), \alpha^2(t))\, X^2(t).$$

For ε small - so as to stay within the domain of S - define

$$\sigma(t, \varepsilon) = S(\alpha^1(t) + X^1(t)\varepsilon,\ \alpha^2(t) + X^2(t)\varepsilon).$$

The requirements are easily checked. QED.

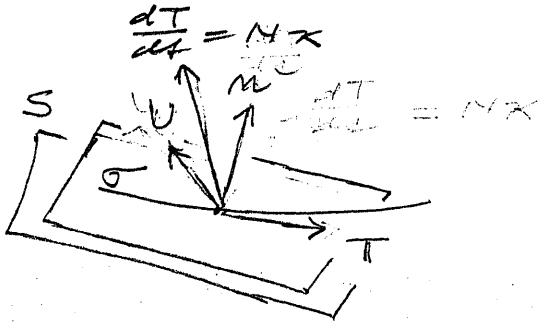
Curvatures of a curve on a surface

Let $\sigma(t)$ be a curve on a surface S, parameterized by arc length t, so

$T = \frac{d\sigma}{ds}(t)$ - unit tangent

$N(t) = N(\sigma(t))$ - unit normal to surface - along σ

$U = N \times T$ - unit vector tangent to surface and orthogonal to the curve.



Note! T, U, N is a right handed orthonormal frame along σ.

1/28/01 Decompose the curvature vector: $\frac{dT}{ds} = N \times$:

$$\frac{dT}{ds}(t) = U(t)\,\kappa_g(t) + N(t)\,\kappa_N(t).$$

We call

$\kappa_g(t) = \frac{dT}{ds} \cdot U$ — geodesic curvature of σ at $\sigma(t)$

$\kappa_N(t) = \frac{dT}{ds} \cdot N$ — normal curvature of σ at $\sigma(t)$.

Later we will see that κ_g is intrinsic - while κ_N is extrinsic.

Shortest paths

Suppose $\sigma_0(s)$ is a curve which minimizes length of curves over all curves joining two prescribed points on a surface S. These points are $\sigma_0(0)$ and $\sigma_0(l)$. Then: It must be that the first variation $\frac{dL}{d\varepsilon}(0)$ is zero for any variation $\sigma(s,\varepsilon)$ of $\sigma_0(s)$ having fixed end points. That is,

$$-\int_0^l \frac{\partial \sigma}{\partial \varepsilon}(s,0)\cdot\frac{\partial^2 \sigma}{\partial s^2}(s,0)\,ds = 0.$$

Now, write the curvature vector of σ_0 as

$$\frac{\partial^2 \sigma_0}{\partial s^2}(s) = U(s)\kappa_g(s) + N(s)\kappa_N(s).$$

The vector field

$$X(s) = U(s)s(l-s)\kappa_g(s)$$

along σ_0 vanishes at $s=0$ and $s=l$. By the lemma, there is a variation $\sigma(s,\varepsilon)$ with variation $\frac{\partial \sigma}{\partial \varepsilon}(s,0) = X(s)$. Then

$$-\int_0^l U\kappa_g s(l-s)\cdot(U\kappa_g + N\kappa_N)\,ds = 0,$$

so

$$\int_0^l s(l-s)\kappa_g(s)^2\,ds = 0.$$

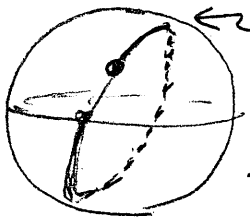
This can happen only if the geodesic curvature $\kappa_g(s)$ is identically zero.

Thus: A curve which minimizes the length of paths between two points satisfies the equivalent conditions:

- the geodesic curvature is zero along the curve
- the acceleration vector is normal to the surface - along the curve
- DEF the curve is a geodesic.

Remark The equivalent conditions are necessary for a shortest path, but not sufficient. A geodesic need not be a shortest path.

Eg

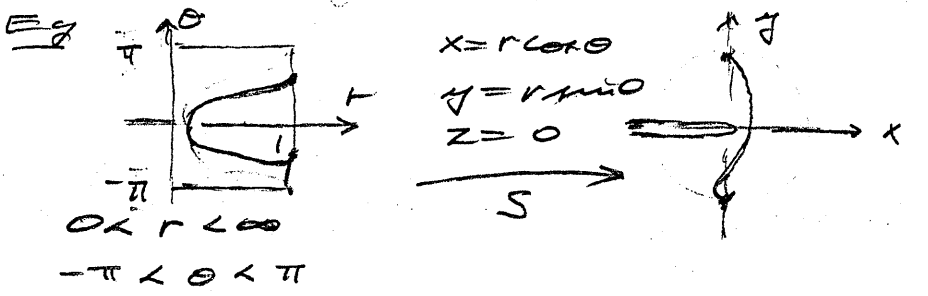


Both "segments" are geodesics, only one is a shortest path.

Problem (6.1) a) Show that for a curve on a surface, $\kappa^2 = \kappa_g^2 + \kappa_n^2$.
and $\kappa_n = \kappa \cos\phi$ (ϕ = curvature) (principal normal and surface normal).
b) Illustrate a) by calculating the various quantities for the circles of latitude on a surface of revolution.

Problem (6.2) How can one handle problems of points "missing" from a surface?

Eg



Acceleration

$\sigma(t) = S(\alpha(t))$ curve on surface S
t not necessarily arc length
s arc length along σ

Velocity vector:

$$V = \frac{d\sigma}{dt} = \frac{d\sigma}{ds}\frac{ds}{dt} = T\frac{ds}{dt}$$

Acceleration:

$$\frac{d^2\sigma}{dt^2} = \frac{dV}{dt} = \frac{d}{dt}\left(\tau \frac{ds}{dt}\right) = \frac{d\tau}{ds}\left(\frac{ds}{dt}\right)^2 + \tau \frac{d^2 s}{dt^2}$$

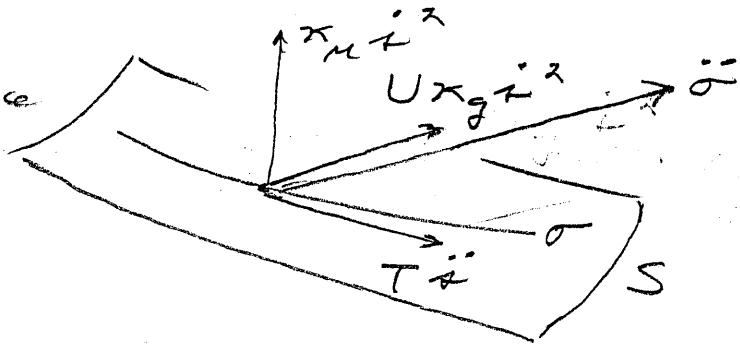
$$= (U\varkappa_g + N\varkappa_N)\left(\frac{ds}{dt}\right)^2 + \tau \frac{d^2 s}{dt^2}$$

so, the acceleration decomposes into:

$$\frac{d^2\sigma}{dt^2} = \underbrace{\tau \frac{d^2 s}{dt^2} + U\varkappa_g \left(\frac{ds}{dt}\right)^2}_{\text{in surface}} + \underbrace{N\varkappa_N \left(\frac{ds}{dt}\right)^2}_{\text{normal to surface}}$$

(along curve; centrifugal)

Remark The component "in" the surface is the intrinsic derivative $\frac{D}{dt}\left(\frac{d\sigma}{dt}\right)$.



For the next two problems and henceforth, we adopt $(X|Y)$ for the inner-product of two vectors with respect to the second fundamental form. If $X = S_1(u_0,v_0)X^1 + S_2(u_0,v_0)X^2$ and $Y = S_1 Y^1 + S_2 Y^2$ are vectors tangent to S at $P_0 = S(u_0,v_0)$, then

$$(X|Y)_{\text{at } P_0} = {}^T\begin{bmatrix} X^1 \\ X^2 \end{bmatrix} \begin{bmatrix} L(u_0,v_0) & M(u_0,v_0) \\ M(u_0,v_0) & N(u_0,v_0) \end{bmatrix} \begin{bmatrix} Y^1 \\ Y^2 \end{bmatrix}.$$

Problem (6.3) Let $\sigma = S\alpha$ be a curve on a surface S, and let

$$V(t) = S_1(\alpha(t))\dot{\alpha}^1(t) + S_2(\alpha(t))\dot{\alpha}^2(t)$$

be its velocity vector.

a) Show $\frac{dV}{dt} = S_{11}(\dot{\alpha}^1)^2 + 2S_{12}\dot{\alpha}^1\dot{\alpha}^2 + S_{22}(\dot{\alpha}^2)^2 + S_1\ddot{\alpha}^1 + S_2\ddot{\alpha}^2$.

b) Show that $\frac{dV}{dt} \cdot N = (V|V)$ at each point of σ.

c) Show that normal curvature κ_N is given by $\kappa_N = \frac{(V|V)}{V \cdot V}$ at each point of σ.

Consequently, Two curves having the same velocity vectors at a point have the same normal curvature.

Problem (6.4) Show that the Frenet equations for the frame T, U, N at a point $S(x(t))$ of a curve on the surface S is:

$$\frac{d}{dt}[T\ U\ N] = [T\ U\ N] \begin{bmatrix} 0 & -\kappa_g & -(T|T) \\ \kappa_g & 0 & (U|T) \\ (T|T) & -(T|U) & 0 \end{bmatrix}.$$