Application Liebmann's theorem.

Fundamental forms and equations with lines of curvature as parameter curves.

$X(u,v)$ regular surface S — $u = \text{const.}$ $v = \text{const.}$ lines of curvature.

$F = 0$ so $ds^2 = E\,du^2 + G\,dv^2$

$\frac{\partial N}{\partial u} = -\varkappa \frac{\partial X}{\partial u}$ $\varkappa = \varkappa_1$ or $\varkappa_2$ (Rodrigues)

so $M = -\frac{\partial X}{\partial v} \cdot \frac{\partial N}{\partial u} = 0.$

and $II = L\,du^2 + N\,dv^2.$

Principal curvatures: $\varkappa_1$ and $\varkappa_2$ are $\frac{L}{E}$ and $\frac{N}{G}$.

Codazzi - Mainardi eqns:

$$\frac{\partial M}{\partial u} - \frac{\partial L}{\partial v} + \Gamma_{12}^1 L + \Gamma_{12}^2 M - \Gamma_{11}^1 M - \Gamma_{11}^2 N = 0$$

$$\frac{\partial N}{\partial u} - \frac{\partial M}{\partial v} + \Gamma_{22}^1 L + \Gamma_{22}^2 M - \Gamma_{21}^1 M - \Gamma_{21}^2 N = 0$$

or

$$0 - \frac{\partial L}{\partial v} + \left(\frac{1}{2}\frac{E_2}{E}\right) L + 0 - 0 - \left(-\frac{1}{2}\frac{E_2}{G}\right) N = 0$$

$$\frac{\partial N}{\partial u} - 0 + \left(-\frac{1}{2}\frac{G_1}{E}\right) L + 0 - 0 - \left(\frac{1}{2}\frac{G_2}{G}\right) N = 0$$

i.e.

$$\frac{\partial L}{\partial v} = \frac{1}{2}\frac{\partial E}{\partial v}\left(\frac{L}{E} + \frac{N}{G}\right) = \frac{1}{2}\frac{\partial E}{\partial v}(\varkappa_1 + \varkappa_2)$$

$$\frac{\partial N}{\partial u} = \frac{1}{2}\frac{\partial G}{\partial u}\left(\frac{L}{E} + \frac{N}{G}\right) = \frac{1}{2}\frac{\partial G}{\partial u}(\varkappa_1 + \varkappa_2).$$

Liebmann's Theorem (1900). The sphere is the only closed regular surface in three-dimensional space having constant Gaussian curvature.

Lemma (Hilbert).

S (piece of) a regular surface with $K > 0$

P_0 point where the larger principal curvature κ_2 attains a max and the smaller κ_1 attains a min.

Then: P_0 is an umbilical point.

Pf. 1) Assume P_0 **not** umbilical so $\kappa_2 > \kappa_1$ near P_0. Introduce lines of curvature as parameters.

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial v}\left(\frac{E_2}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_1}{\sqrt{EG}}\right)\right)$$

$$= -\frac{1}{2}\left(\frac{E_{22}}{EG} - \frac{1}{2}\,\frac{E_2^2 G + E_2 G_2 E}{E^2 G^2} + \frac{G_{11}}{EG} - \frac{1}{2}\,\frac{G_1^2 E + E_1 G_1 G}{E^2 G^2}\right)$$

$$\frac{\partial L}{\partial v} = \frac{1}{2}\frac{\partial E}{\partial v}(\kappa_1 + \kappa_2), \quad \frac{\partial N}{\partial u} = \frac{1}{2}\frac{\partial G}{\partial u}(\kappa_1 + \kappa_2).$$

$$\kappa_2 = \frac{L}{E}, \quad \kappa_1 = \frac{N}{G} \quad \text{(interchange } u \text{ and } v \text{ if necessary)}.$$

So: $L = E\kappa_2$, $\quad \frac{\partial L}{\partial v} = \frac{\partial E}{\partial v}\kappa_2 + E\frac{\partial \kappa_2}{\partial v}$

$N = G\kappa_1$; $\quad \frac{\partial N}{\partial u} = \frac{\partial G}{\partial u}\kappa_1 + G\frac{\partial \kappa_1}{\partial u}$.

$$\frac{\partial L}{\partial v} = \frac{\partial E}{\partial v} x_2 + E \frac{\partial x_2}{\partial v} = \frac{1}{2} \frac{\partial E}{\partial v} (x_1 + x_2)$$

or $$\frac{\partial E}{\partial v} = - \frac{2E}{x_2 - x_1} \frac{\partial x_2}{\partial v}.$$

Similarly, $\frac{\partial G}{\partial u} = \frac{2G}{x_2 - x_1} \frac{\partial x_1}{\partial u}$

2) Substitute into equation for K:

$$-2EGK = - \frac{2E}{x_2 - x_1} \frac{\partial^2 x_2}{\partial v^2} + \frac{2G}{x_2 - x_1} \frac{\partial^2 x_1}{\partial u^2}$$

$$+ g(u,v) \frac{\partial x_2}{\partial v} + f(u,v) \frac{\partial x_1}{\partial u}$$

bounded functions

At P_0:

$$\left.\begin{aligned} \frac{\partial x_2}{\partial v} &= 0 \\ \frac{\partial^2 x_2}{\partial v^2} &\leq 0 \end{aligned}\right\} \; x_2 \text{ a maximum.}$$

$$\left.\begin{aligned} \frac{\partial x_1}{\partial u} &= 0 \\ \frac{\partial^2 x_1}{\partial u^2} &\geq 0 \end{aligned}\right\} \; x_1 \text{ a minimum.}$$

Thus at P_0:

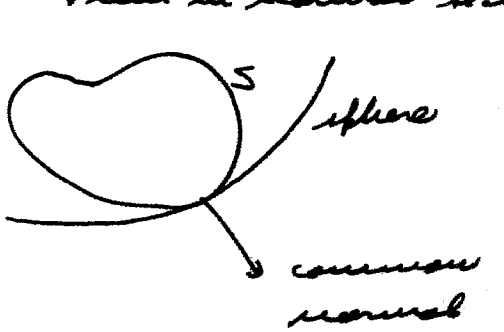
$$\underbrace{-2EGK}_{<0} = \underbrace{- \frac{2E}{x_2 - x_1} \frac{\partial^2 x_2}{\partial v^2}}_{\geq 0} + \underbrace{\frac{2G}{x_2 - x_1} \frac{\partial^2 x_1}{\partial u^2}}_{\geq 0}$$

Contradiction. QED.

Proof of Liebmann's theorem.

1) S closed surface of constant curvature K.

Enclose S in a sphere;



at tangent point, compare curvatures of normal sections with common normal. $K > 0$ at this point, so $K > 0$ on S.

2) $\varkappa_1, \varkappa_2$ the smaller and larger principal curvatures at each point. $\varkappa_1, \varkappa_2$ continuous fcns on S ($\varkappa_1, \varkappa_2 = H \pm \sqrt{H^2 - K}$).

P_0 : point where $\varkappa_2$ attains max

Then: ~~[illegible]~~ $\varkappa_1(P_0) =$ min

max $\varkappa_1 = \dfrac{K}{\varkappa_2}$.

Hilbert lemma: P_0 is umbilical : $\varkappa_1(P_0) = \varkappa_2(P_0)$.

If $P \in S$,

$$\varkappa_2(P_0) \geq \varkappa_2(P) \geq \varkappa_1(P) \geq \varkappa_1(P_0)$$

(the two outer terms are equal)

so $\varkappa_2(P) = \varkappa_1(P)$. Every point of S is an umbilical point. S is a sphere. QED.

i.e. A soap bubble can not be beat.