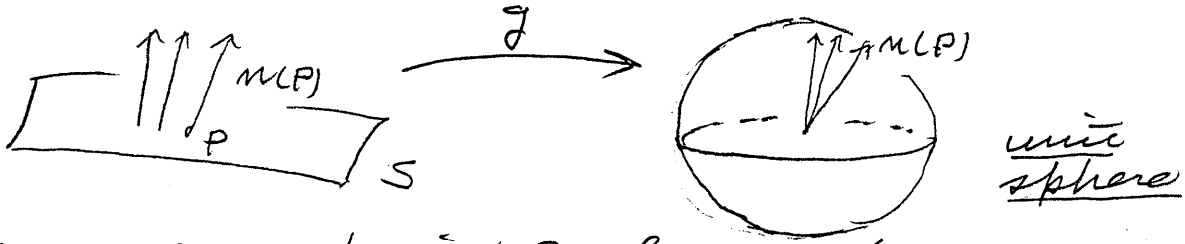
5 Surfaces in space. The Gauss map.

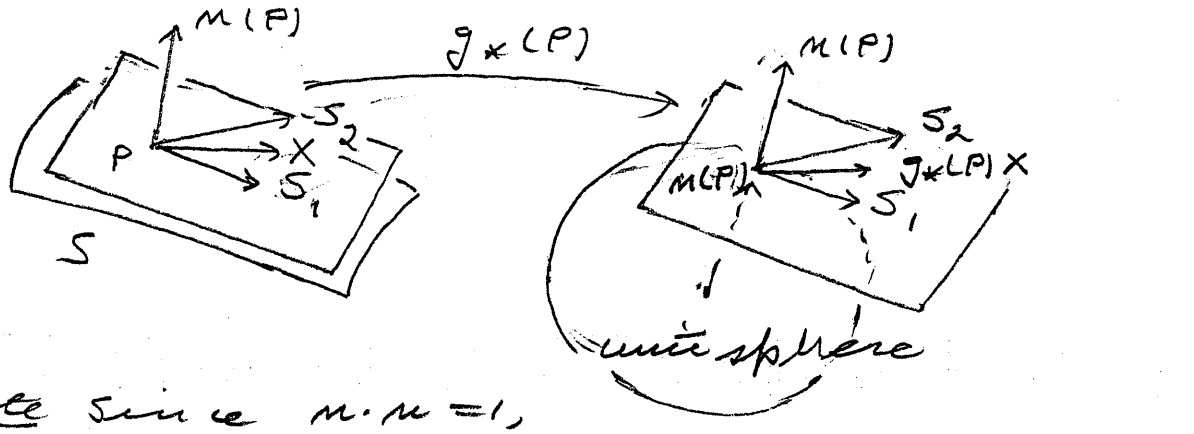
Gauss map



g sends a point P of a surface S to the point N(P) on the unit sphere. The vector N(P) at P on S is "translated" to the center of the unit sphere.

Derivative of the Gauss map.

$$g_*(P): \begin{pmatrix} \text{tangent} \\ \text{vectors} \\ \text{to } S \text{ at } P \end{pmatrix} \longrightarrow \begin{pmatrix} \text{tangent} \\ \text{vectors to unit} \\ \text{sphere at } N(P) \end{pmatrix}$$



Note Since $N \cdot N = 1$, the tangent plane to the unit sphere at N(P) has normal N(P). Since this is in Euclidean E^3, we may "translate" vectors and "identify" these tangent spaces. Especially, $g_*(P)$ may be viewed as a linear transformation of one space to itself.

Recall If $X = S_1(u_0, v_0) X^1 + S_2(u_0, v_0) X^2$, then $g_*(S(u_0, v_0) X$ is obtained by taking

any curve $\alpha(t)$ having $\alpha(t_0) = (u_0, v_0)$ and velocity vector $\frac{d}{dt} S(\alpha(t))\Big|_{t=t_0} = X$, and computing

$$g_*(S(u_0,v_0))X = \frac{d}{dt} g(S(\alpha(t)))\Big|_{t=t_0}$$

The computation

1) Fix a point $P_0 = S(u_0, v_0)$ on S. Take as basis vectors

$$S_1 = \frac{\partial S}{\partial u}(u_0, v_0) \text{ and } S_2 = \frac{\partial S}{\partial v}(u_0, v_0)$$

2) Now

$$g_*(P_0) S_1 = g_*(S(u_0,v_0)) \left(\frac{d}{du} S(u, v_0)\Big|_{u=u_0} \right)$$

$$= \frac{d}{du} g(S(u, v_0))\Big|_{u=u_0} = \frac{d}{du} N(S(u, v_0))\Big|_{u=u_0}$$

$$= \frac{\partial N}{\partial u}(S(u_0, v_0)) = N_1(P_0)$$

Likewise

$$g_*(P_0) S_2 = N_2(P_0).$$

3) Let W (2x2) be the matrix of $g_*(P_0)$ with respect to the basis S_1 and S_2:

$$g_*(P_0)[S_1\ S_2] = [S_1\ S_2] W$$

$$[N_1\ N_2] = [S_1\ S_2] W.$$

Multiply by ${}^t[S_1\ S_2]$ on left:

$${}^t[S_1\ S_2] \cdot [N_1\ N_2] = {}^t[S_1\ S_2] \cdot [S_1\ S_2] W$$

$$\begin{bmatrix} S_1 \cdot N_1 & S_1 \cdot N_2 \\ S_2 \cdot N_1 & S_2 \cdot N_2 \end{bmatrix} = \begin{bmatrix} S_1 \cdot S_1 & S_1 \cdot S_2 \\ S_2 \cdot S_1 & S_2 \cdot S_2 \end{bmatrix} W$$

$$\begin{bmatrix} -L & -M \\ -M & -N \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix} W,$$

so

$$W = -\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

= matrix of $g_*(P_0)$ in basis S_1 and S_2.

This holds at any point P

Weingarten equations From 3) above we have

$$[N_1\ N_2] = -[S_1\ S_2]\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}\begin{bmatrix} L & M \\ M & N \end{bmatrix}$$

Expanding gives

$$N_1 = S_1\frac{FM-GL}{EG-F^2} + S_2\frac{FN-GM}{EG-F^2}$$

$$N_2 = S_1\frac{FL-EM}{EG-F^2} + S_2\frac{FM-EN}{EG-F^2}$$

Problem (5.1) Show that the second fundamental form is given by

$$\mathrm{II}(X) = -X\cdot g_*(P)X \quad \text{at } P.$$

Curvature $\det\left(-\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}\begin{bmatrix} L & M \\ M & N \end{bmatrix}\right) = \dfrac{LN-M^2}{EG-F^2}$

so the Gauss curvature is the determinant of g_*:

$$K(P) = \det g_*(P).$$

2/23/01 Area From $[N_1\ N_2] = [S_1\ S_2]W$ we have

$$N_1 \times N_2 = (S_1 \times S_2)\det W$$

so

$$\underbrace{\|N_1 \times N_2\|\,du\,dv}_{\text{area element } d\omega \text{ on unit sphere}} = \det W\ \underbrace{\|S_1 \times S_2\|\,du\,dv}_{dS}$$

Thus

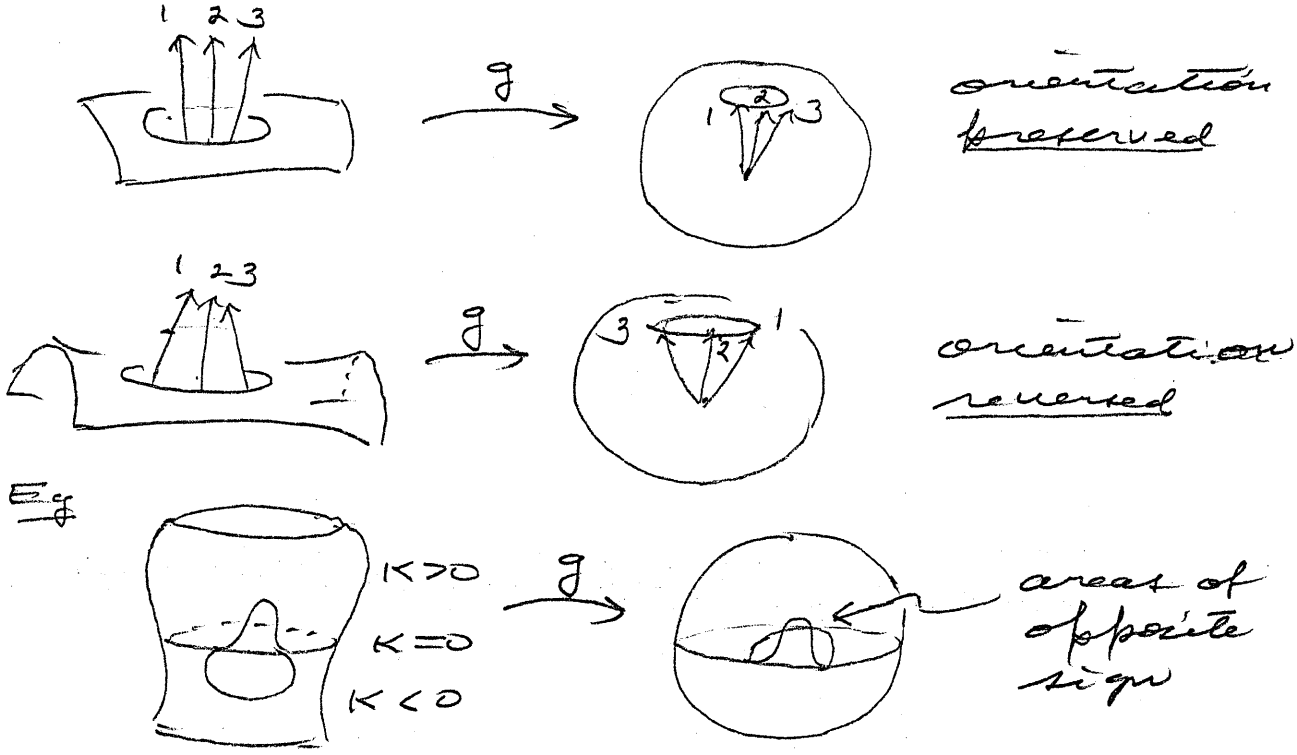
$$d\omega = K\,dS$$

– area element is "signed area" of orientable surface

and

$$\int_R K\,dS = \int_{g(R)} d\omega = \left(\begin{array}{l}\text{area of } g(R) \text{ on unit sphere counted with overlaps \& signs}\end{array}\right)$$

Note Larger $K(P)$ means g "spreads out" the surface more near P — with signs.



Problem (5.2) a) Show that mean curvature is the trace of the derivative of the Gauss map: $H(P) = \frac{1}{2}\operatorname{trace} g_*(P)$.
b) Show that the map $-g_*(P)$ has as its eigenvalues and eigenvectors the principal curvatures and principal directions at P.

Problem (5.3) Let S be the torus of revolution (Prob. 4.9), and let R be the region of S where the Gauss curvature K is positive. Find

$$\int_R K\,dS \quad \text{and} \quad \int_{S\setminus R} K\,dS.$$

Interpret in terms of the spherical image of R and $S\setminus R$ under g.

Independance on parameters

We learned from problems 3.3 & 4.2 that when a surface is parameterized in two ways

$X = S(u,v)$ and $X = S'(u',v')$,

the first and second fundamental forms change by

$$\begin{bmatrix} E' & F' \\ F' & G' \end{bmatrix} = {}^{\tau}J \begin{bmatrix} E & F \\ F & G \end{bmatrix} J \quad , \quad J = \begin{bmatrix} \frac{\partial \Phi^1}{\partial u'} & \frac{\partial \Phi^1}{\partial v'} \\ \frac{\partial \Phi^2}{\partial u'} & \frac{\partial \Phi^2}{\partial v'} \end{bmatrix},$$

at (u',v') at $\Phi(u',v')$

where the change of parameters is

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \Phi^1(u',v') \\ \Phi^2(u',v') \end{bmatrix},$$

and

$$\begin{bmatrix} L' & M' \\ M' & N' \end{bmatrix} = {}^{\tau}J \begin{bmatrix} L & M \\ M & N \end{bmatrix} J.$$

Then

$$-\begin{bmatrix} E' & F' \\ F' & G' \end{bmatrix}^{-1} \begin{bmatrix} L' & M' \\ M' & N' \end{bmatrix} = J^{-1} \left(-\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix} \right) J.$$

The determinant and trace of similar matrices are equal, so one has $K' = K$ and $H' = H$ for the Gauss and mean curvature.

But, the Gauss map g is defined without reference to coördinates. The derivative $g_*(P)$ is a linear transformation - defined without use of coördinates or basis. The determinant and trace of $g_*(P)$ are thus defined geometrically - without reference to coördinates. Use of any coördinates will yield valid formulas for K & H. I.e. One knows *a priori* that these are "well-defined" - independant of coördinates.

Note K and H are still extrinsic - dependant on the surface being in Euclidean space.

Problem (5.4) Show that with the change of parameters $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \Phi^1(u',v') \\ \Phi^2(u',v') \end{bmatrix}$ above, one has

$$E'G' - F'^2 = (EG - F^2)(\det J)^2$$

and

$$du'\,dv' = |\det J|\,du\,dv.$$

Hence the element of surface area dS is

$$\sqrt{E'G' - F'^2}\,du'\,dv' = \sqrt{EG - F^2}\,du\,dv,$$

and independent of parameters.

Aside: $N' = N \cdot \operatorname{sgn} J$ to incorporate the orientation of the unit normals.

The element of surface area dS is not only independent of parameters, it is also intrinsic - does not depend on being a surface in space. This is reflected in dS being expressed in terms of E, F, G alone and not needing L, M, N.

Gauss Theorema Egregium

Theorem The Gauss curvature K is intrinsic - it depends only on the first fundamental form $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$.

One says K is a "bending invariant".

Goal: A formula for K in terms of E, F, G and derivatives only. There are several.

Preliminary - determinantal identities

Set

$$A = \begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}, \quad B = \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix}, \quad X = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}, \quad Y = \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix}$$

Then

$$A \cdot X \times Y = \begin{vmatrix} a^1 & x^1 & y^1 \\ a^2 & x^2 & y^2 \\ a^3 & x^3 & y^3 \end{vmatrix}$$

$$= a^1 \begin{vmatrix} x^2 & y^2 \\ x^3 & y^3 \end{vmatrix} - a^2 \begin{vmatrix} x^1 & y^1 \\ x^3 & y^3 \end{vmatrix} + a^3 \begin{vmatrix} x^1 & y^1 \\ x^2 & y^2 \end{vmatrix}$$

and

$$(A \cdot X \times Y)(B \cdot X \times Y) = \begin{vmatrix} a^1 & x^1 & y^1 \\ a^2 & x^2 & y^2 \\ a^3 & x^3 & y^3 \end{vmatrix} \begin{vmatrix} b^1 & x^1 & y^1 \\ b^2 & x^2 & y^2 \\ b^3 & x^3 & y^3 \end{vmatrix}$$

$$= \begin{vmatrix} A\cdot B & A\cdot X & A\cdot Y \\ X\cdot B & X\cdot X & X\cdot Y \\ Y\cdot B & Y\cdot X & Y\cdot Y \end{vmatrix}.$$

transpose this determinant

Proof by Baltzer Beginning

$$K = \frac{LN - M^2}{EG - F^2}, \quad L_{ij} = S_{ij} \cdot N = \frac{S_{ij} \cdot S_1 \times S_2}{\sqrt{EG - F^2}},$$

$\frac{\partial^2 S}{\partial u^i \partial u^j}$

so

$$K = \frac{(S_{11} \cdot S_1 \times S_2)(S_{22} \cdot S_1 \times S_2) - (S_{12} \cdot S_1 \times S_2)^2}{(EG - F^2)^2}.$$

We will express all numerator terms in terms of E, F, G and their derivatives.

Now, $(EG - F^2)K$

$$= \begin{vmatrix} S_{11}\cdot S_{22} & S_{11}\cdot S_1 & S_{11}\cdot S_2 \\ S_1 \cdot S_{22} & E & F \\ S_2 \cdot S_{22} & F & G \end{vmatrix} - \begin{vmatrix} S_{12}\cdot S_{12} & S_{12}\cdot S_1 & S_{12}\cdot S_2 \\ S_1 \cdot S_{12} & E & F \\ S_2 \cdot S_{12} & F & G \end{vmatrix}$$

$$= S_{11}\cdot S_{22} \begin{vmatrix} E & F \\ F & G \end{vmatrix} + \begin{vmatrix} 0 & S_{11}\cdot S_1 & S_{11}\cdot S_2 \\ S_{22}\cdot S_1 & E & F \\ S_{22}\cdot S_2 & F & G \end{vmatrix}$$

$$- S_{12}\cdot S_{12} \begin{vmatrix} E & F \\ F & G \end{vmatrix} - \begin{vmatrix} 0 & S_{12}\cdot S_1 & S_{12}\cdot S_2 \\ S_{12}\cdot S_1 & E & F \\ S_{12}\cdot S_2 & F & G \end{vmatrix}$$

DETERMINANT OF COEFFICIENTS

$$= (S_{11}\cdot S_{22} - S_{12}\cdot S_{12}) \begin{vmatrix} E & F \\ F & G \end{vmatrix} + \left(\begin{matrix} \text{two} \\ \text{determinants} \\ \text{above} \end{matrix} \right)$$

By differentiation of

$$E = S_1 \cdot S_1 , \quad F = S_1 \cdot S_2 , \quad G = S_2 \cdot S_2$$

obtain:

$$S_{11}\cdot S_1 = \tfrac{1}{2} E_1 \qquad S_{11}\cdot S_2 = F_1 - \tfrac{1}{2} E_2$$

$$S_{12}\cdot S_1 = \tfrac{1}{2} E_2 \qquad S_{12}\cdot S_2 = \tfrac{1}{2} G_1$$

$$S_{22}\cdot S_1 = F_2 - \tfrac{1}{2} G_1 \qquad S_{22}\cdot S_2 = \tfrac{1}{2} G_2$$

where

$$E_1 = \frac{\partial E}{\partial u} , \text{ etc.}$$

For example,

$$F_2 = \frac{\partial}{\partial v}(S_1 \cdot S_2) = S_{12}\cdot S_2 + S_1 \cdot S_{22} = \tfrac{1}{2} G_1 + S_1 \cdot S_{22}$$

From $S_{11}\cdot S_2 = F_1 - \tfrac{1}{2} E_2$

obtain

$$S_{112}\cdot S_2 + S_{11}\cdot S_{22} = F_{12} - \tfrac{1}{2} E_{22} .$$

From $S_{12}\cdot S_2 = \tfrac{1}{2} G_1$

obtain $S_{121}\cdot S_2 + S_{12}\cdot S_{21} = \tfrac{1}{2} G_{11}$.

Subtract:

$$S_{11}\cdot S_{22} - S_{12}\cdot S_{12} = -\tfrac{1}{2} G_{11} + F_{12} - \tfrac{1}{2} E_{22}$$

All quantities are now in terms of E, F, G and their first and second derivatives which completes the proof.

Final formula

$$K = \frac{-\frac{1}{2} G_{11} + F_{12} - \frac{1}{2} E_{22}}{EG - F^2} + \frac{1}{(EG-F^2)^2} \begin{vmatrix} 0 & \frac{1}{2} E_1 & F_1 - \frac{1}{2} E_2 \\ F_2 - \frac{1}{2} G_1 & E & F \\ \frac{1}{2} G_2 & F & G \end{vmatrix}$$

$$- \frac{1}{(EG-F^2)^2} \begin{vmatrix} 0 & \frac{1}{2} E_2 & \frac{1}{2} G_1 \\ \frac{1}{2} E_2 & E & F \\ \frac{1}{2} G_1 & F & G \end{vmatrix} ,$$

Problem (5.5) If the parameter curves are orthogonal, so $F(u,v)=0$ all u, v, then

$$K = -\frac{1}{\sqrt{EG}}\left(\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right).$$

– a "divergence" expression for K.

Example For "geodesic polar coördinates" where $ds^2 = dr^2 + G(r,\theta)\,d\theta^2$, one has

$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial r^2}.$$

Eg For a surface of revolution, $ds^2 = dr^2 + g(r)^2\,d\theta^2$ and $K = -\frac{g''}{g}$.