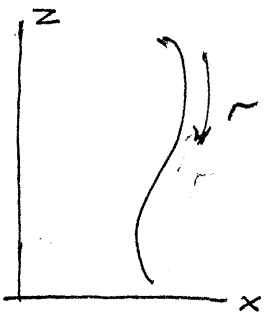
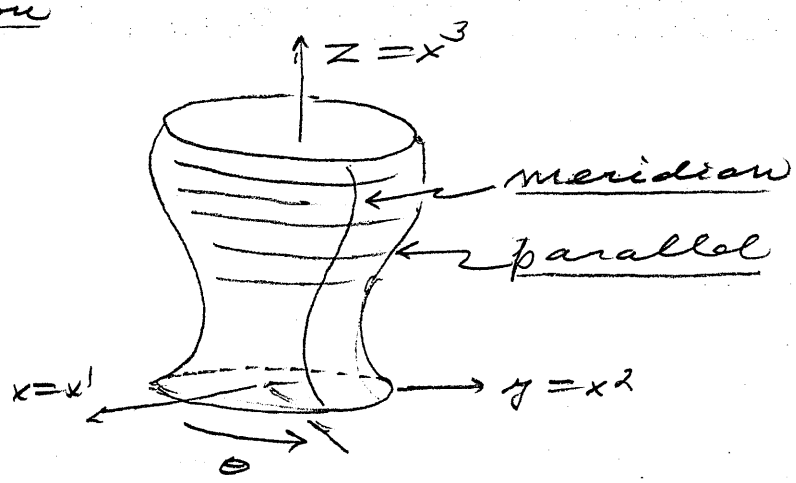
Example Surfaces of revolution

Parameterization



"profile curve" parameterized by arc length r.



$$\begin{cases} x = g(r) \\ z = h(r) \end{cases} \qquad g' = \frac{dg}{dr} \qquad h' = \frac{dh}{dr} \qquad (g')^2 + (h')^2 = 1$$

since r is arc length.

Surface of revolution:

$$X = S(r,\theta) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} g(r) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} h(r) = \begin{bmatrix} g(r)\cos\theta \\ g(r)\sin\theta \\ h(r) \end{bmatrix}$$

First fundamental form

$$\frac{\partial S}{\partial r} = \begin{bmatrix} g'(r)\cos\theta \\ g'(r)\sin\theta \\ h'(r) \end{bmatrix}, \qquad \frac{\partial S}{\partial \theta} = \begin{bmatrix} -g(r)\sin\theta \\ g(r)\cos\theta \\ 0 \end{bmatrix}$$

$$E(r,\theta) = \frac{\partial S}{\partial r} \cdot \frac{\partial S}{\partial r} = (g')^2 + (h')^2 = 1$$

$$F = \frac{\partial S}{\partial r} \cdot \frac{\partial S}{\partial \theta} = 0 \quad \text{– coördinate curves are orthogonal}$$

$$G = \frac{\partial S}{\partial \theta} \cdot \frac{\partial S}{\partial \theta} = g^2 \quad \text{– function of } r \text{ only}$$

so

$$ds^2 = dr^2 + g(r)^2 d\theta^2.$$

Area element

$$\frac{\partial S}{\partial r} \times \frac{\partial S}{\partial \theta} = \begin{vmatrix} \hat{i} & g'\cos\theta & -g\sin\theta \\ \hat{j} & g'\sin\theta & g\cos\theta \\ \hat{k} & h' & 0 \end{vmatrix} = \begin{bmatrix} -gh'\cos\theta \\ -gh'\sin\theta \\ gg' \end{bmatrix}$$

$$\left\| \frac{\partial S}{\partial r} \times \frac{\partial S}{\partial \theta} \right\|^2 = (-gh')^2 + (gg')^2 = g^2 \underbrace{(g'^2 + h'^2)}_{1} = g^2$$

$$\left\| \frac{\partial S}{\partial r} \times \frac{\partial S}{\partial \theta} \right\| = \sqrt{EG - F^2} = g$$

so the area element is

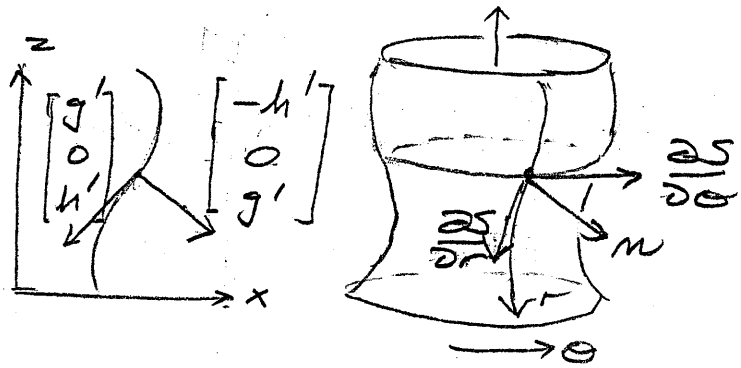
$$dS = g(r)\, dr\, d\theta$$

Aside The Euclidean plane in a system of revolution in polar coördinates. The profile curve is the x-axis:

$x = r$, $z = 0$ so $g(r) = r$, $h(r) = 0$. $\quad ds^2 = dr^2 + r^2 d\theta^2$, $\quad dS = r\, dr\, d\theta$.

Normal vector

n points "inward" if the profile curve is traversed "downward."



$$n(r,\theta) = \frac{\frac{\partial S}{\partial r} \times \frac{\partial S}{\partial \theta}}{\left\| \frac{\partial S}{\partial r} \times \frac{\partial S}{\partial \theta} \right\|} = \begin{bmatrix} -h'(r)\cos\theta \\ -h'(r)\sin\theta \\ g'(r) \end{bmatrix}$$

Second fundamental form

$$\frac{\partial^2 S}{\partial r^2} = \begin{bmatrix} g''(r)\cos\theta \\ g''(r)\sin\theta \\ h''(r) \end{bmatrix}, \quad L(r,\theta) = \frac{\partial^2 S}{\partial r^2} \cdot n = -g''h' + g'h''$$

$$\frac{\partial^2 S}{\partial r \partial \theta} = \begin{bmatrix} -g'(r)\sin\theta \\ g'(r)\cos\theta \\ 0 \end{bmatrix}, \quad \Pi = \frac{\partial^2 S}{\partial r \partial \theta} \cdot n = 0$$

$$\frac{\partial^2 S}{\partial \theta^2} = \begin{bmatrix} -g(r)\cos\theta \\ -g(r)\sin\theta \\ 0 \end{bmatrix}, \quad N = \frac{\partial^2 S}{\partial \theta} \cdot n = gh'$$

so

$$II = (-g''(r)h'(r) + g'(r)h''(r))\, dr^2 + g(r)h'(r)\, d\theta^2$$

Principal curvatures

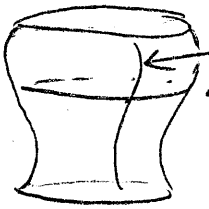
$$\begin{bmatrix} L-\lambda E & M-\lambda F \\ M-\lambda F & N-\lambda G \end{bmatrix} = \begin{bmatrix} (-g''h'+g'h'')-\lambda & 0 \\ 0 & gh'-\lambda g^2 \end{bmatrix}$$

has eigenvalues

$$\kappa_1 = g'h'' - g''h' \qquad \text{(These may not be in increasing order.)}$$

$$\kappa_2 = \frac{h'}{g}$$

The principal directions are $\frac{\partial S}{\partial r}$ and $\frac{\partial S}{\partial \theta}$.



Aside Note that

$$\kappa_1 = \frac{d}{dr}\operatorname{Arctan}\frac{h'}{g'} \qquad \left(\text{Use } r = \text{arc length and } (g')^2+(h')^2 = 1.\right)$$

is the curvature of the profile curve (with sign).

Gauss curvature

$$K = \frac{LN-M^2}{EG-F^2} = \kappa_1\kappa_2 = \frac{(-g''h'+g'h'')gh'}{g^2}$$

$$= \frac{(g'h''-g''h')h'}{g} = \frac{g'(h'h'')-g'(h')^2}{g}$$

Use $(g')^2+(h')^2=1$ to obtain $g'g''+h'h''=0$.

Then

$$K = \frac{g'(-g'g'')-g'(h')^2}{g} = -\frac{g''}{g}\underbrace{\left((g')^2+(h')^2\right)}_{1}$$

so

$$K(r,\theta) = -\frac{g''(r)}{g(r)}$$

Or

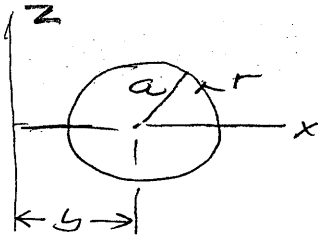
$$K = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial r^2}$$

Note K is intrinsic - it depends on the first fundamental form only.

This formula holds in general in geodesic polar coordinates: $ds^2 = dr^2 + G(r,\theta)\,d\theta^2$ on any surface.

2/16/01

Example Torus

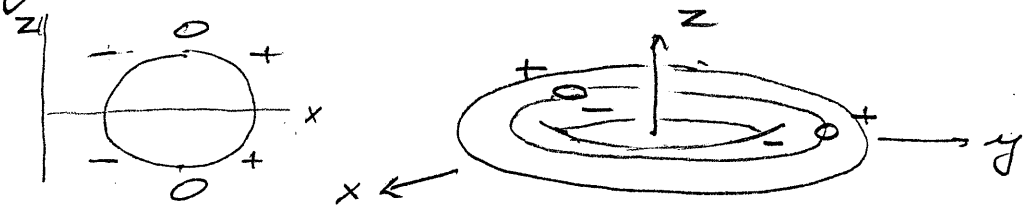


$$\begin{cases} x = g(r) = b + a\cos\frac{r}{a} \\ z = h(r) = a\sin\frac{r}{a} \end{cases} \qquad 0 < a < b$$

r = arc length on circle of radius a

$$K = \pm\frac{g''}{g} = -\frac{-\frac{1}{a}\cos\frac{r}{a}}{b + a\cos\frac{r}{a}} = \frac{1}{a^2}\,\frac{\cos\frac{r}{a}}{\frac{b}{a} + \cos\frac{r}{a}}$$

Signs:



Problem (4.9) For the torus above, show:

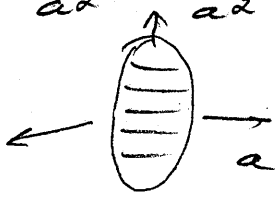
a) $$\frac{1}{a^2}\,\frac{-1}{\frac{b}{a} - 1} \le K \le \frac{1}{a^2}\,\frac{1}{\frac{b}{a} + 1}$$

b) $$K = \frac{1}{a^2}\,\frac{\sqrt{x^2+y^2} - b}{\sqrt{x^2+y^2}}$$

Problem (4.10) Spheroid $\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$



$a > c$ prolate



$a < c$ oblate

a) Show that arc lengths along the ellipse $\begin{cases} x = a\sin\tau \\ z = c\cos\tau \end{cases}$ (τ is not an angle) is given by

$$s = a\int_0^{\tau}\sqrt{1 - e^2\sin^2\tau}\,d\tau$$

where $e = \sqrt{1 - \frac{c^2}{a^2}}$ $(c < a)$ is the eccentricity of the ellipse.

b) An <u>elliptic integral</u> of the <u>second kind</u> is the function

$$F(k,\tau) = \int_0^{\tau} \sqrt{1-k^2\sin^2\tau}\, d\tau$$

where $-1 < k < 1$ is a parameter called the <u>modulus</u>. Write $\tau = T(k,u)$ for the inverse of $u = F(k,\tau)$. Show that $\frac{dT}{du} = \frac{1}{\sqrt{1-k^2\sin^2 T(k,u)}}$.

c) Show that the ellipse is parameterized in terms of its arc length s by

$$\begin{cases} x = a\sin T(e, \frac{s}{a}) \\ z = c\cos T(e, \frac{s}{a}) \end{cases}.$$

d) Using $g(r) = a\sin T(e, \frac{r}{a})$ and $K = -g''/g$, show that the curvature of the (prolate) spheroid is

$$K = \frac{1}{a^2}\,\frac{1-e^2}{(1-e^2\sin^2 T(e,\frac{r}{a}))^2}.$$

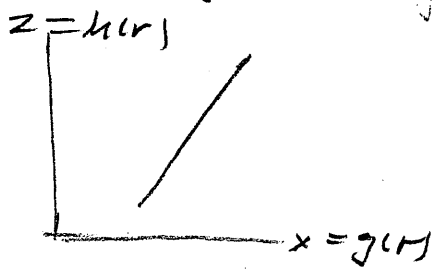
e) If $e = 0$, $F(0,\tau) = \tau$, $T(0,u) = u$, and the spheroid is a sphere of radius $a = c$. In this case $K = \frac{1}{a^2}$.

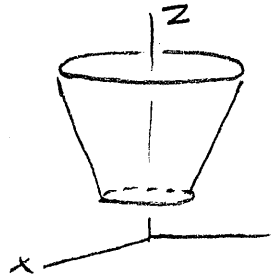
<u>Constant curvature</u> - surfaces of revolution

<u>Zero</u> $K = 0$ all r, θ.

From $-\frac{g''}{g} = 0$ obtain $g(r) = ar + b$,

from $(g')^2 + (h')^2 = 1$ obtain $h(r) = \sqrt{1-a^2} + c$





Right circular cone

<u>Note</u>! Plane is special case $a = -1$, $b = 0$

Remark The right circular cylinder also has $K=0$. Obtain this from $\kappa_1 = 0$. (since g = CONSTANT).

Positive $K = +1$ all r, θ - prototype

From $-\frac{g''}{g} = 1$ or $g'' + g = 0$

(obtain $g(r) = C\sin(r - r_0)$ C = constant, r_0 = constant

We may assume $r_0 = 0$;

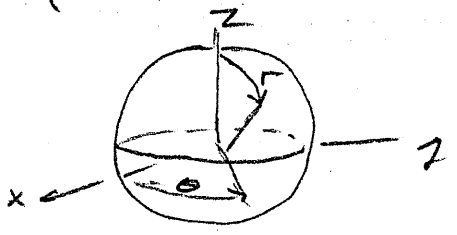
$$g(r) = C\sin r$$

$$h(r) = \int_0^r \sqrt{1 - C^2\cos^2\rho}\, d\rho \quad \text{by } (g')^2 + (h')^2 = 1.$$

i) Case $C = 1$

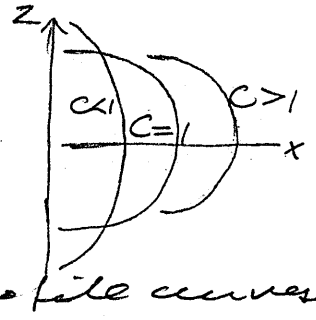
$$\begin{cases} g(r) = \sin r \\ h(r) = \cos r \end{cases}$$

$$S(r,\theta) = \begin{bmatrix} \sin r\cos\theta \\ \sin r\sin\theta \\ \cos r \end{bmatrix}$$



Unit sphere in spherical coördinates
r = colatitude
$ds^2 = dr^2 + \sin^2 r\, d\theta^2$

ii) Other C



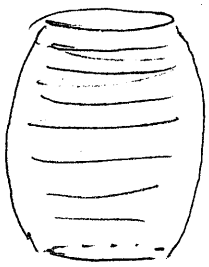
Profile curves

$$S(r,\theta;C) = \begin{bmatrix} C\sin r\cos\theta \\ C\sin r\sin\theta \\ \int_0^r \sqrt{1 - C^2\cos^2\rho}\, d\rho \end{bmatrix}$$

These surfaces are all locally isometric with the unit sphere! $r' = r$, $\theta' = C\theta$;

$ds^2 = dr^2 + C^2\sin^2 r\, d\theta^2$
$= dr'^2 + \sin^2 r'\, d\theta'^2$.

A region cut out from a unit sphere can be bent without tearing or stretching.



$C < 1$

Problem (4.11). Find any restrictions on r, θ, C necessary for the bending of a portion of the unit sphere. Cut out a tennis ball to demonstrate your assertions.

Negative $K = -1$ all r, θ – prototype

From $-\frac{g''}{g} = -1$ or $g'' - g = 0$

obtain

$$g(r) = Ae^{r} + Be^{-r} \qquad \begin{matrix} A = \text{constant} \\ B = \text{constant} \end{matrix}$$

$$h(r) = \int_0^r \sqrt{1 - (Ae^{\rho} + Be^{-\rho})^2}\, d\rho + \text{constant}$$

CONST

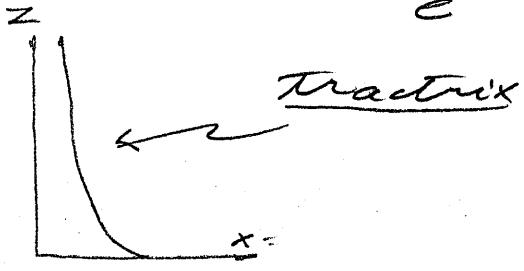
i) Special case: Pseudosphere

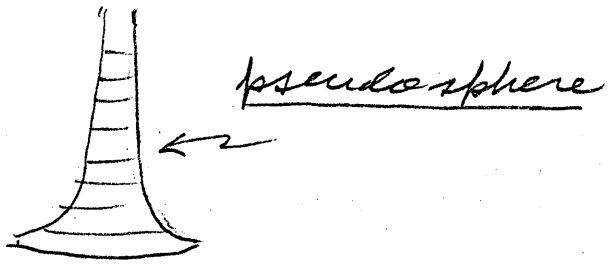
$$g(r) = e^{-r}$$

$$h(r) = \int_0^r \sqrt{1 - (-e^{-\rho})^2}\, d\rho$$

$$= \ln \frac{1 + \sqrt{1 - e^{-2r}}}{1 - \sqrt{1 - e^{-2r}}} - r - \sqrt{1 - e^{-2r}}$$

$$= \ln \frac{1 + \sqrt{1 - e^{-2r}}}{e^{-r}} - \sqrt{1 - e^{-2r}}$$





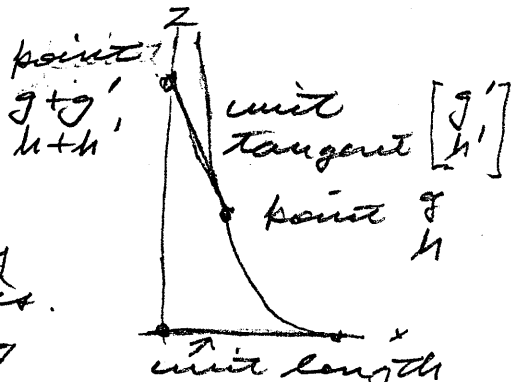
Remark There are other pseudospherical surfaces corresponding to other A and B.

Aside The tractrix is a drag curve:

$$g + g' = 0$$

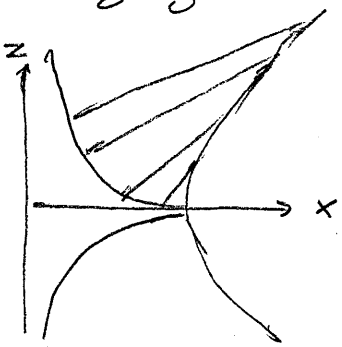
$$h + h' = \ln \frac{e^{-r}}{1 - \sqrt{1 - e^{-2r}}} - r.$$

The unit tangent at any point meets the vertical axis. (Tangent is dragged along



Remark The pseudosphere has finite area $\int dS = \int_0^{\infty}\int_0^{2\pi} g(r)\, d\theta\, dr = \int_0^{\infty}\int_0^{2\pi} e^{-r}\, d\theta\, dr = 2\pi.$

Problem (4.12) The "evolute" of a planar curve is the locus of its centers of curvature. Show that the evolute of the tractrix above is the catenary $x = \cosh z$. The tractrix may be obtained from the catenary as the locus of the end of a string as it "unwinds" from the catenary. See! Any reference for a discussion of evolutes and involutes will generally do.



Problem (4.13) The surface of revolution given by

$$S(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ a\ln\left(r+\sqrt{r^2-a^2}\right) \end{bmatrix}$$

is generated by revolving a catenary about its axis.

a) Show the surface S is a minimal surface - the mean curvature H is zero.

b) Show that this surface is the only minimal surface of revolution.