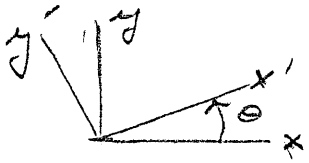
2/5/01 **4. Surfaces in space. Second fundamental form.**

Quadratic surfaces

Quadratic surface in space:

$z = Ax^2 + 2Bxy + Cy^2$ "quadratic form"

or $z = {}^t\begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} A & B \\ B & C \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ A, B, C constants



Rotate through θ in xy-plane:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^{-1}\begin{bmatrix} x \\ y \end{bmatrix}$$

Obtain $z = A'x'^2 + 2B'x'y' + C'y'^2$

where

$$\begin{bmatrix} A' & B' \\ B' & C' \end{bmatrix} = {}^t\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} A & B \\ B & C \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Especially

$A' + C' = A + C$ trace
$A'C' - B'^2 = AC - B^2$ det
} invariants

and $B' = (C-A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta)$

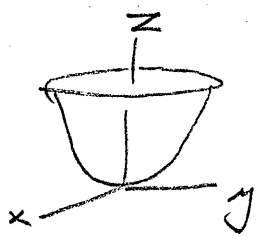
Choose θ so $\tan 2\theta = \frac{2B}{C-A}$; then $B' = 0$.

Surface has form: $z = a_1x^2 + a_2y^2$, $a_1 \le a_2$,
- omitting primes

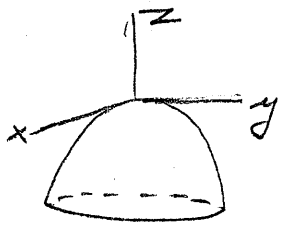
Cases

Elliptic paraboloid

$AC - B^2 = a_1a_2 > 0$



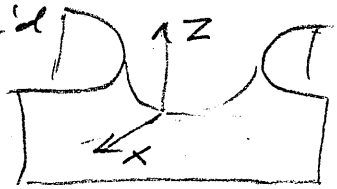
$a_1 > 0, a_2 > 0$
positive definite



$a_1 < 0, a_2 < 0$
negative definite

Hyperbolic paraboloid

$AC - B^2 = a_1a_2 < 0$

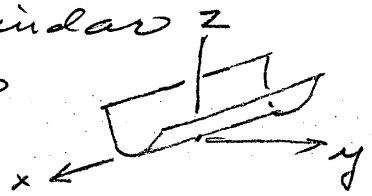


$a_1 < 0 < a_2$

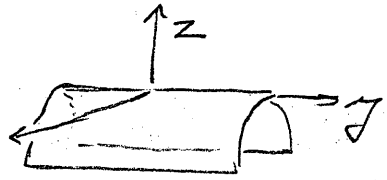
Parabolic cylinder

$AC - B^2 = a_1 a_2 = 0$

rank = 1



$a_1 = 0,\ a_2 > 0$

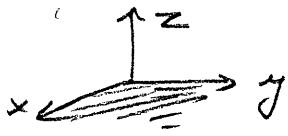


$a_1 < 0,\ a_2 = 0$

Plane

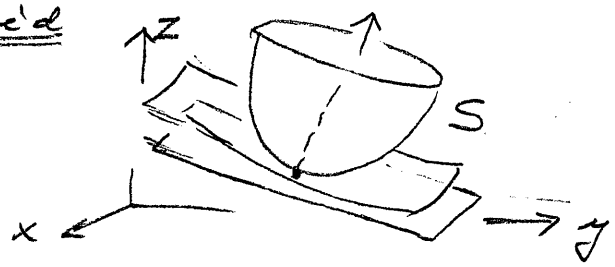
rank = 0

$a_1 = 0,\ a_2 = 0$



Osculating paraboloid

The "best fitting" quadratic surface - as measured from the tangent plane at $S(u,v)$ (fixed u and v) is the osculating paraboloid at $S(u,v)$.



Taylor expansion at $S(u,v)$:

$$S(u+h, v+k) = S(u,v) + S_1 h + S_2 k + \frac{1}{2}(S_{11} h^2 + 2S_{12} hk + S_{22} k^2) + O(3)$$

where

$S_1 = \frac{\partial S}{\partial u}(u,v)$

$S_{11} = \frac{\partial^2 S}{\partial u^2}(u,v)$, etc.

S_1 & S_2 span tangent vectors at $S(u,v)$.

Unit normal to tangent plane:

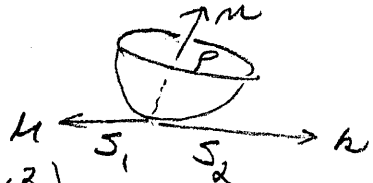
$$n(u,v) = \frac{S_1 \times S_2}{\|S_1 \times S_2\|}$$

Distance of surface from tangent plane - at $S(u,v)$!

$$\text{DIST} = (S(u+h, v+k) - S(u,v)) \cdot n(u,v)$$

$$= \frac{1}{2}(S_{11} \cdot n\, h^2 + 2S_{12} \cdot n\, hk + S_{22} \cdot n\, k^2) + O(3)$$

Osculating paraboloid referred to tangent plane:



$$P = \frac{1}{2}(S_{11} \cdot n\, h^2 + 2S_{12} \cdot n\, hk + S_{22} \cdot n\, k^2)$$

where $S_{11}\cdot N$ is $\frac{\partial^2 S}{\partial u^2}(u,v)\cdot N(u,v)$, etc.
This gives the "shape" of the surface near $S(u,v)$. Depends on u & v.

Second fundamental form

Quadratic form on the tangent space to surface S at $S(u,v)$:

$$II = L(u,v)\,du^2 + 2M(u,v)\,du\,dv + N(u,v)\,dv^2$$

where - at $S(u,v)$:

$$L = S_{11}\cdot N,\quad M = S_{12}\cdot N,\quad N = S_{22}\cdot N$$

or

$$II = \sum_{i,j=1}^{2} L_{ij}(u^1,u^2)\,du^i\,du^j$$

where

$$L_{ij}(u^1,u^2) = S_{ij}(u^1,u^2)\cdot N(u^1,u^2) = \frac{\partial^2 S}{\partial u^i \partial u^j}\cdot N.$$

We have

$$L_{11} = L,\quad L_{12} = L_{21} = M,\quad L_{22} = N.$$

Note $\frac{\partial S}{\partial u^i}\cdot N = 0$ since $\frac{\partial S}{\partial u^i} = S_i$ is a tangent vector. Then

$$\frac{\partial^2 S}{\partial u^i \partial u^j}\cdot N + \frac{\partial S}{\partial u^i}\cdot\frac{\partial N}{\partial u^j} = \frac{\partial}{\partial u^j}\left(\frac{\partial S}{\partial u^i}\cdot N\right) = 0$$

so

$$L_{ij} = -\frac{\partial S}{\partial u^i}\cdot\frac{\partial N}{\partial u^j}.$$

or

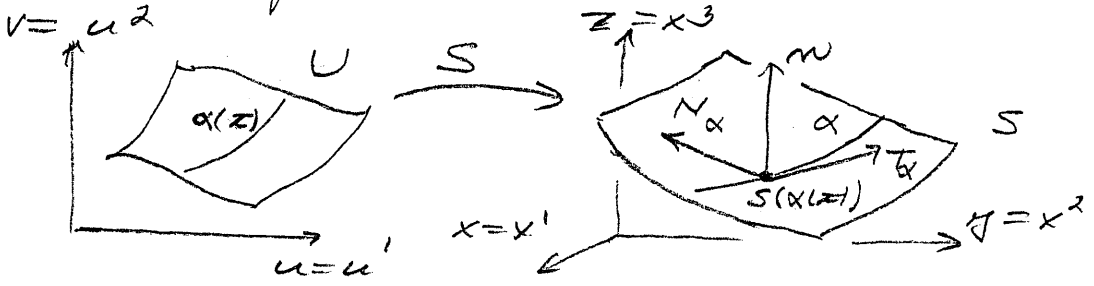
$$L = -\frac{\partial S}{\partial u}\cdot\frac{\partial N}{\partial u}$$

$$M = -\frac{\partial S}{\partial u}\cdot\frac{\partial N}{\partial v} = -\frac{\partial S}{\partial v}\cdot\frac{\partial N}{\partial u}$$

$$N = -\frac{\partial S}{\partial v}\cdot\frac{\partial N}{\partial v} \qquad \text{at } S(u,v).$$

2/7/01

Curvature of curves on a surface



$X = S(\alpha(t))$ curve on S

$\frac{d}{dt} S(\alpha(t)) = T_\alpha(s) \frac{ds}{dt}(t)$ velocity vector

T_α = unit tangent to $S \circ \alpha$

s = arc length along $S \circ \alpha$

$\frac{d}{dt} S(\alpha(t)) = S_1(\alpha(t))\dot{\alpha}^1(t) + S_2(\alpha(t))\dot{\alpha}^2(t)$

velocity vector in basis S_1, S_2.

Differentiate again:

$$N_\alpha \kappa_\alpha \dot{s}^2 + T_\alpha \ddot{s} = S_{11}(\dot{\alpha}^1)^2 + 2S_{12}\dot{\alpha}^1\dot{\alpha}^2 + S_{22}(\dot{\alpha}^2)^2 + S_1\ddot{\alpha}^1 + S_2\ddot{\alpha}^2 \quad \text{at } S(u^1, u^2).$$

Dot with $N(u^1, u^2)$ - the unit normal to S at $S(u^1, u^2)$. Observe that T_α, S_1, S_2 are tangent to S. (N_α need not be tangent to S.)

$$(N_\alpha \cdot N)\kappa_\alpha \dot{s}^2 = (S_{11}\cdot N)(\dot{\alpha}^1)^2 + 2(S_{12}\cdot N)\dot{\alpha}^1\dot{\alpha}^2 + (S_{22}\cdot N)(\dot{\alpha}^2)^2.$$

Let

θ_α = angle between N_α and N

= (angle between principal normal to $S \circ \alpha$ and normal to S)

- at $S(\alpha(t))$. Depends on point (on t).

Then, using $L = S_{11}\cdot N$, $M = S_{12}\cdot N$, $N = S_{22}\cdot N$ and $\dot{s}^2 = E(\dot{\alpha}^1)^2 + 2F\dot{\alpha}^1\dot{\alpha}^2 + G(\dot{\alpha}^2)^2$ we have:

$$\kappa_\alpha \cos\theta_\alpha = \frac{L(\dot{\alpha}^1)^2 + 2M\dot{\alpha}^1\dot{\alpha}^2 + N(\dot{\alpha}^2)^2}{E(\dot{\alpha}^1)^2 + 2F\dot{\alpha}^1\dot{\alpha}^2 + G(\dot{\alpha}^2)^2},$$

Note The right-hand side depends only on the tangent vector

$$T_\alpha = \frac{S_1\dot{\alpha}^1 + S_2\dot{\alpha}^2}{\dot{s}} = S_1\alpha'^1 + S_2\alpha'^2$$

to the curve $S \circ \alpha$ at $S(\alpha(t))$.

Consequence Meusnier's Theorem. 1776. All curves having the same tangent and principal normal at a point of a surface have the same curvature at that point provided that the principal normals are not tangent to the surface.

Remarks 1) The last condition insures that $\cos\theta = N_\alpha \cdot N$ is not zero,
2) $\kappa\cos\theta$ is the "normal curvature" of the curve - to arise later in conjunction with geodesics. It is the component of "$S(\alpha(t))$" normal to the surface.

Problem (4.1) Find expressions for $L(x,y)$, $M(x,y)$, $N(x,y)$ for a surface in Monge form $z = f(x,y)$.

Problem (4.2) A point $S(u,v)$ on the surface S is called elliptic, parabolic or hyperbolic according as $LN - M^2$ is >0, $=0$, or <0. Show that when the parameters are changed from u,v to u',v' the quantity $LN - M^2$ is multiplied by $\left|\frac{D(u,v)}{D(u',v')}\right|^2$. Hence, the notions of elliptic, parabolic, and hyperbolic point do not depend on the parameterization.

Problem (4.3) Show that the torus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos u \\ \sin u \\ 0 \end{bmatrix}(R + r\cos v) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}(r \sin v),$$

$0 < r < R$, contains all three types of points.

Problem (4.4) Find a non-trivial example to illustrate Meusnier's Theorem.

Suggestion: The space curve

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a(1+\cos t) \\ a \sin t \\ 2a \sin \frac{t}{2} \end{bmatrix}$$

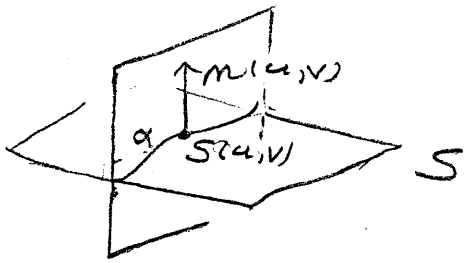
lies on two surfaces:

sphere: $x^2 + y^2 + z^2 = 4a^2$

cylinder: $(x-a)^2 + y^2 = a^2$.

Normal sections

Let $X = S(u,v)$ be a regular surface. Intersect S with the plane passing through $S(u,v)$ (u & v fixed) and "containing" the normal $N(u,v)$.



Claim The intersection is a regular curve near $S(u,v)$.

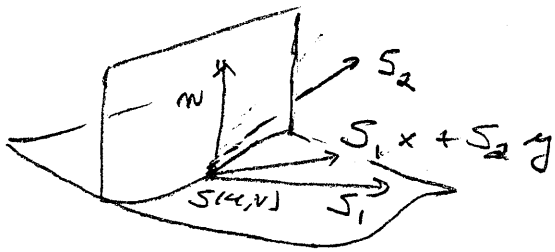
For We may assume that the surface is in Monge form $X = {}^t[x, y, f(x,y)]$ with the point in question being $x=0, y=0$ and the tangent plane the xy-plane. (by translation & rotation of coördinates). Then f, $\partial f/\partial x$, $\partial f/\partial y$ vanish at $x=0, y=0$. Then S meets the plane "spanned" by

$$\begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ in } \alpha(\theta) = \begin{bmatrix} t\cos\theta \\ t\sin\theta \\ f(t\cos\theta, t\sin\theta) \end{bmatrix}.$$

Then $\dot\alpha(0) = {}^t[\cos\theta, \sin\theta, -]$, so $\|\dot\alpha(0)\| \geq 1$. QED.

Curvature

Consider the normal sections determined by intersecting the plane containing $N(u,v)$ and $S_1(u,v)x + S_2(u,v)y$ with the surface S. (x and y are not x^1, x^2 here.)

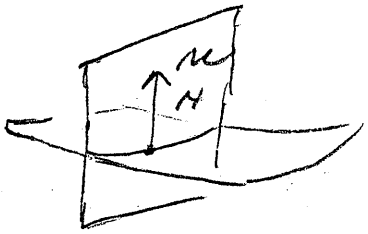


The normal to the curve, N, is in this plane and $\perp$ normal to S, so $N = \pm n$ at $S(u,v)$. Thus, the curvature κ of this normal section is

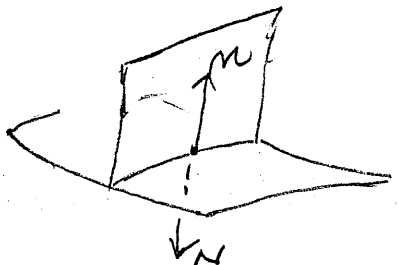
$$\pm\kappa = \frac{L x^2 + 2M xy + N y^2}{E x^2 + 2F xy + G y^2}, \qquad L(u,v), \dots, G(u,v)$$

- etc.

Note: This depends only on L, M, N, E, F, G at $S(u,v)$ and on the direction of the x,y vector $S_1(u,v)\,x + S_2(u,v)\,y$. (a result of two homogeneous forms of same degree.)



- use "+" sign



- use "−" sign

Once the choice of unit normal n is fixed, the expression $Lx^2 + \dots / Ex^2 + \dots$ will give all κ as x and y vary.

Define:

$$\kappa_1(u,v) = \min_{(x,y)\neq(0,0)} \frac{L x^2 + 2M xy + N y^2}{E x^2 + 2F xy + G y^2}$$

$$\kappa_2(u,v) = \max_{(x,y)\neq(0,0)} \frac{\text{— ditto —}}{\text{— ditto —}}$$

where $L, \dots, G$ depend on u & v.

κ_1 & κ_2 principal curvatures of S at $S(u,v)$

$K = \kappa_1 \kappa_2$ Gauss curvature —— " ——

$H = \frac{1}{2}(\kappa_1 + \kappa_2)$ mean curvature —— " ——

$K(u,v)$ and $H(u,v)$ depend on u & v.

Next: A method to compute these.

2/9/01 **Digression** - **the extrema of the ratio of quadratic forms**

Setting: E, F, G, L, M, N fixed scalars
The form $Ex^2 + 2Fxy + Gy^2$ positive definite.
Let $f(x,y) = \dfrac{Lx^2 + 2Mxy + Ny^2}{Ex^2 + 2Fxy + Gy^2}$, $(x,y) \neq (0,0)$

To find:

$x_1 = \min f(x,y)$, $x_2 = \max f(x,y)$
over $(x,y) \neq (0,0)$. Since f is homogeneous of degree 0, we may assume $Ex^2 + 2Fxy + Gy^2 = 1$ and treat this as extrema with constraints:

$$\begin{cases} Lx^2 + 2Mxy + Ny^2 = \min \text{ or } \max \\ Ex^2 + 2Fxy + Gy^2 = 1 \end{cases}$$

Use Lagrange multipliers:

$$\frac{\partial}{\partial x}(Lx^2 + 2Mxy + Ny^2) = \lambda \frac{\partial}{\partial x}(Ex^2 + 2Fxy + Gy^2)$$

$$\frac{\partial}{\partial y}(\text{---}''\text{---}) = \lambda \frac{\partial}{\partial y}(\text{---}''\text{---})$$

to obtain:

$$\begin{cases} (L - \lambda E)x + (M - \lambda F)y = 0 \\ (M - \lambda F)x + (N - \lambda G)y = 0 \end{cases}$$

Facts - easy to prove -

- The roots of $\det \begin{bmatrix} L - \lambda E & M - \lambda F \\ M - \lambda F & N - \lambda G \end{bmatrix} = 0$ are the extrema x_1 and x_2. These are real eigenvalues.
- If $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ are the eigenvectors for x_1 and x_2 i.e. if

$$\begin{bmatrix} L - x_j E & M - x_j F \\ M - x_j F & N - x_j G \end{bmatrix} \begin{bmatrix} x_j \\ y_j \end{bmatrix} = 0, \quad j = 1, 2$$

then

$$\begin{cases} Lx_j^2 + 2Mx_j y_j + Ny_j^2 = x_j & j = 1,2 \\ Ex_j^2 + 2Fx_j y_j + Gy_j^2 = 1 \end{cases}$$

- the last equation is a "normalization"

- If $\kappa_1 \neq \kappa_2$, then the eigenvectors are orthogonal - with respect to both quadratic forms:
$$\begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0. \quad \text{Likewise } L, M, N.$$
- If $\kappa_1 = \kappa_2$, then $f(x,y)$ is a constant and that constant is: $\kappa_1 = \kappa_2 = \frac{L}{E} = \frac{M}{F} = \frac{N}{G}$.

And: the equation $\det \begin{bmatrix} L-\lambda E & M-\lambda F \\ M-\lambda F & N-\lambda G \end{bmatrix} = 0$ expands to
$$\lambda^2 - \frac{EN-2FM+GL}{EG-F^2}\lambda + \frac{LN-M^2}{EG-F^2} = 0.$$
The coefficients are the sum & product of roots κ_1 and κ_2:
$$\kappa_1 + \kappa_2 = \frac{EN-2FM+GL}{EG-F^2}, \quad \kappa_1\kappa_2 = \frac{LN-M^2}{EG-F^2}.$$

Consequence. For a surface $X = S(u,v)$ with coefficients $E(u,v), \dots, L(u,v)$ of the first and second fundamental forms, we have:
$$K = \frac{LN-M^2}{EG-F^2}, \quad H = \frac{1}{2}\frac{EN-2FM+GL}{EG-F^2}.$$
Gauss curvature, mean curvature - at $S(u,v)$. These depend on u & v.

Remarks 1) Both K and H appear to depend on the second fundamental form. Later: K can be computed from E, F, G (and derivatives) alone. The Gauss curvature K is, in fact, intrinsic.
2) A surface is a minimal surface when its mean curvature H vanishes identically.

Classification of points on a surface

1) $K > 0$ elliptic point
   i) $\kappa_1 = \kappa_2$ umbilic
   - characterized by $\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$
2) $K = 0$
   i) not all of L, M, N are zero - parabolic point
   ii) all three of L, M, N are zero - planar point
3) $K < 0$ hyperbolic point

These terms reflect the shape of the osculating paraboloid at a point. The osculating paraboloid yields more - as in Euler & Dupin to follow.

Principal directions At a point $S(u,v)$ on a surface S, the eigenvectors associated with $\kappa_1(u,v)$ and $\kappa_2(u,v)$ give the principal directions. Denote the unit vectors by $E_1(u,v)$ and $E_2(u,v)$. These vector are tangent to S but need not be $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ of computing. Fix u and v and write every vector X tangent to S at $S(u,v)$ as

$$X = X^1 S_1(u,v) + S_2(u,v) X^2, \quad S_1 = \frac{\partial S}{\partial u}, \; S_2 = \frac{\partial S}{\partial v}.$$

Likewise express Y. Then

$$X \cdot Y = \begin{bmatrix} X^1 \\ X^2 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} Y^1 \\ Y^2 \end{bmatrix}.$$

Especially

$$X \cdot X = E(X^1)^2 + 2FX^1X^2 + G(X^2)^2$$

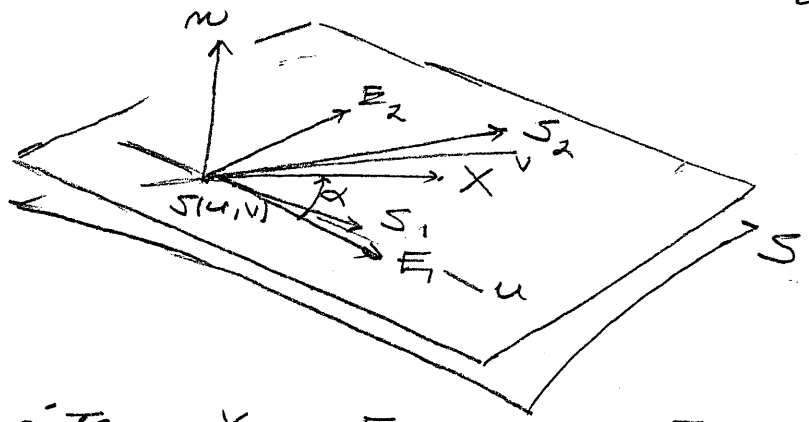
$$\mathrm{II}(X) = L(X^1)^2 + 2MX^1X^2 + N(X^2)^2.$$

The curvature of normal sections appear as $X \cdot X / \mathrm{II}(X)$.

Remark Principal directions at an umbilic or planar point are indeterminant.

Euler's Theorem setting as above.
Use E_1, E_2 at $S(u,v)$ as a basis for the tangent space.

$$S_1 = \frac{\partial S}{\partial u}, \quad S_2 = \frac{\partial S}{\partial v}$$



Write $X = E_1 \cos\alpha + E_2 \sin\alpha$
for an arbitrary unit vector tangent to S at $S(u,v)$. Then

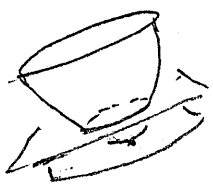
$X \cdot X = 1$ - unit vector

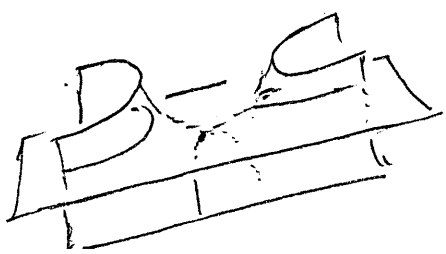
$II(X) = II(E_1)\cos^2\alpha + II(E_2)\sin^2\alpha$
- orthogonality with respect to II.

Since $II(E_1) = \kappa_1$ and $II(E_2) = \kappa_2$,
the curvature of the normal section obtained by intersecting S with the plane of N and X is

$$\pm\kappa = \kappa_1 \cos^2\alpha + \kappa_2 \sin^2\alpha.$$ Euler's theorem.

2/12/01 Dupin's indicatrix The intersection of the osculating paraboloid with a plane parallel to the tangent plane to a surface at a point gives information about the surface near that point.





At the point $S(u,v)$ (u & v fixed) of a surface S, the Dupin indicatrix is given by $\mathrm{II}(X) = 1$, a conic. Here X ranges over all vectors tangent to S at $S(u,v)$. We have in terms of:

- basis $S_1 = \frac{\partial S}{\partial u}$ and $S_2 = \frac{\partial S}{\partial v}$ at $S(u,v)$

$$X = S_1 X^1 + S_2 X^2$$

$$L(X^1)^2 + 2M X^1 X^2 + N(X^2)^2 = 1$$

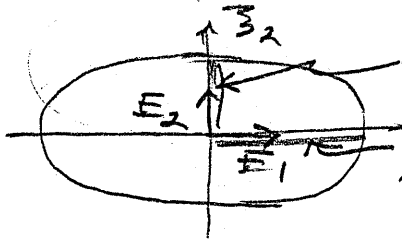
- basis E_1 and E_2 of unit vectors in the principal directions at $S(u,v)$:

$$X = E_1 (\bar{3}^1)^2 + E_2 (\bar{3}^2)^2$$

$$\kappa_1 (\bar{3}^1)^2 + \kappa_2 (\bar{3}^2)^2 = 1 \quad \text{by Euler}$$

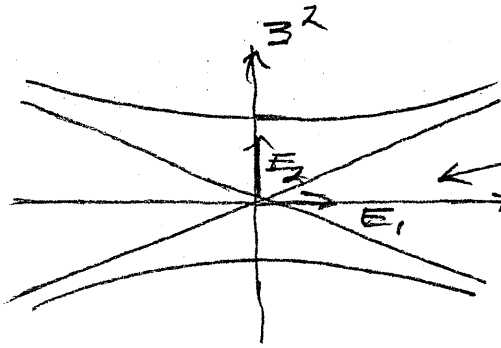
Typical conics are:

elliptic point $\qquad 0 < \kappa_1 \leq \kappa_2$



$$\frac{(\bar{3}^1)^2}{\left(\frac{1}{\sqrt{\kappa_1}}\right)^2} + \frac{(\bar{3}^2)^2}{\left(\frac{1}{\sqrt{\kappa_2}}\right)^2} = 1$$

hyperbolic point $\qquad \kappa_1 < 0 < \kappa_2$



$$-\frac{(\bar{3}^1)^2}{\left(\frac{1}{\sqrt{-\kappa_1}}\right)^2} + \frac{(\bar{3}^2)^2}{\left(\frac{1}{\sqrt{\kappa_2}}\right)^2} = 1$$

Problem (4.5) Find the umbilics of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and prove that tangent planes at these points are parallel to planes which intersect the ellipsoid in circles.

Lines of curvature

A curve on a surface is a <u>line of curvature</u> if its tangent at each point is a principal direction. They may be obtained from an O.D.E.

<u>Digression</u> – continued.
Setting: E, F, G, L, M, N fixed scalars – as before. Let ${}^t[x\ y]$ be an eigenvector giving a principal direction, and λ the corresponding eigenvalue

$$\begin{bmatrix} L-\lambda E & M-\lambda F \\ M-\lambda F & N-\lambda G \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Expand and eliminate λ to obtain:

$$\frac{Lx+My}{Ex+Fy} = \frac{Mx+Ny}{Fx+Gy}.$$

Expand, cancel terms, and write as a determinant!

$$\begin{vmatrix} y^2 & -xy & x^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0.$$

<u>Consequence</u> One has a quadratic ODE for the lines of curvature on a parametrized surface $X = S(u,v)$:

$$\begin{vmatrix} dv^2 & -du\,dv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0, \qquad E = E(u,v), \text{ etc.}$$

<u>Problem</u> (4.6) a) Find the equations of a line of curvature for a surface in Monge form $z = f(x,y)$.
b) For the surface $z = xy$, the equation for lines of curvature is integrable by elementary functions. Use this to parametrize the surface by lines of curvature.
<u>Ans</u> $x = \sinh(u+v)$, $y = \cosh(u-v)$, $z = xy$.

Remark Principal directions are orthogonal, so parameterizing a surface by lines of curvature gives a system of "orthogonal coördinates". One must avoid umbilics and planar points.

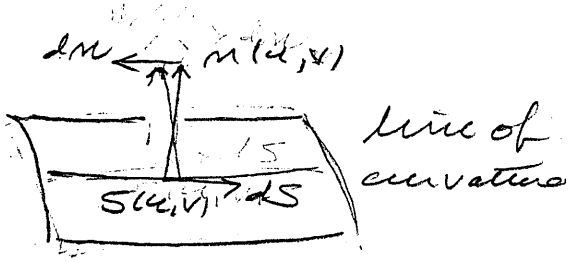
Remark The confocal quadrics: $\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} + \frac{z^2}{c^2-\lambda} = 1$ with $a^2 < b^2 < c^2$ consist of three families: ellipsoids, hyperboloids of one sheet, and hyperboloids of two sheets. Quadrics of different families intersect in curves which are lines of curvature in each. This gives a triply orthogonal system of surfaces for coördinates in space. Ref Struik, Ch. 2, Para. 2-11 and 2-6.

Problem (4.7) Show that the parameter curves $S(u, \text{const})$ and $S(\text{const}, v)$ are lines of curvature if and only if both M and F are identically zero.

Rodrigues's formula - a characterization of lines of curvature by the change in $N(u,v)$ being proportional to the change in $S(u,v)$:

$$dN + \kappa\, dS = 0$$

along a line of curvature having principal curvature $\kappa(u,v)$.



Proof A line of curvature $X = S(\alpha(t))$ is characterized by

$$\frac{d}{dt} S(\alpha(t)) = S_1(\alpha(t))\dot{\alpha}^1(t) + S_2(\alpha(t))\dot{\alpha}^2(t)$$

being an eigenvector:

$$\begin{bmatrix} L - \kappa E & M - \kappa F \\ M - \kappa F & N - \kappa G \end{bmatrix} \begin{bmatrix} \dot{\alpha}^1 \\ \dot{\alpha}^2 \end{bmatrix} = 0$$

or

$$\sum_{j=1}^{2} (L_{ij} - \kappa g_{ij})\dot{\alpha}^j = 0 \quad , \quad i = 1,2,$$

$\leftarrow L_{ij}(\alpha(t))$, etc

Use $L_{ij} = -\frac{\partial S}{\partial u^i} \cdot \frac{\partial N}{\partial u^j}$ and $g_{ij} = \frac{\partial S}{\partial u^i} \cdot \frac{\partial S}{\partial u^j}$ to obtain

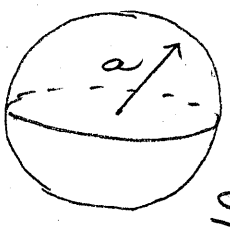
$$\sum_{j=1}^{2} \left(-\frac{\partial S}{\partial u^i} \cdot \frac{\partial N}{\partial u^j} - \kappa \frac{\partial S}{\partial u^i} \cdot \frac{\partial S}{\partial u^j} \right) \dot{\alpha}^j = 0 \qquad i = 1,2$$

This is

$$-\frac{\partial S}{\partial u^i}(\alpha(t)) \cdot \left(\frac{d}{dt} N(\alpha(t)) + \kappa(\alpha(t)) \frac{d}{dt} S(\alpha(t)) \right) = 0$$

for $i = 1,2$. Now, $\frac{dN}{dt}$ and $\frac{dS}{dt}$ are in the plane of S_1 and S_2; S_1 and S_2 are a basis, so $\frac{dN}{dt} + \kappa \frac{dS}{dt} = 0$ on the curve. QED.

A characterization of the sphere



On a sphere of radius a, $\kappa_1 = \kappa_2 = \frac{1}{a}$. Since every normal section is a circle of radius a. Every point is an umbilic.

Converse: A regular surface, every point of which is an umbilic, is part of a sphere.

Note: If the surface is "closed" – compact, connected, and without boundary – the surface is a sphere.

Proof 1) Every point $S(u,v)$ on the surface S is an umbilic, i.e. $\kappa_1(u,v) = \kappa_2(u,v)$. We may assume these are >0 and write $\kappa(u,v)$ for κ_1 and κ_2. | TO SHOW: κ = CONST.

2) Every curve on S is a line of curvature. Especially the coördinate curves v = constant and u = constant: by Rodrigues.

$$\begin{cases} \frac{\partial N}{\partial u} + \kappa \frac{\partial S}{\partial u} = 0 \\ \frac{\partial N}{\partial v} + \kappa \frac{\partial S}{\partial v} = 0 \end{cases}$$

Note that $\kappa(u,v)$ is differentiable. Differentiate the equations above!

$$\frac{\partial}{\partial v}\frac{\partial N}{\partial u} = -\frac{\partial \kappa}{\partial v}\frac{\partial S}{\partial u} - \kappa \frac{\partial}{\partial v}\frac{\partial S}{\partial u}$$

$$\frac{\partial}{\partial u}\frac{\partial N}{\partial v} = -\frac{\partial \kappa}{\partial u}\frac{\partial S}{\partial v} - \kappa \frac{\partial}{\partial u}\frac{\partial S}{\partial v}$$

Since $\frac{\partial}{\partial v}\frac{\partial}{\partial u} = \frac{\partial}{\partial u}\frac{\partial}{\partial v}$, we have

$$\frac{\partial \kappa}{\partial v}\frac{\partial S}{\partial u} = \frac{\partial \kappa}{\partial u}\frac{\partial S}{\partial v}.$$

Since $\frac{\partial S}{\partial u}$ and $\frac{\partial S}{\partial v}$ are independent, we have

$$\frac{\partial \kappa}{\partial u} = 0 \ \& \ \frac{\partial \kappa}{\partial v} = 0, \text{ so } \kappa(u,v) = \text{CONSTANT} = \kappa.$$

3) From equations of Rodrigues:

$$\frac{\partial N}{\partial u} + \kappa \frac{\partial S}{\partial u} = \frac{\partial}{\partial u}(N + \kappa S) = 0$$

$$\frac{\partial N}{\partial v} + \kappa \frac{\partial S}{\partial v} = \frac{\partial}{\partial v}(N + \kappa S) = 0$$

so $N + \kappa S = \text{CONSTANT VECTOR} = A$

Thus $\left\| S - \frac{1}{\kappa} A \right\| = \frac{1}{\kappa^2}$.

so $S(u,v)$ lies on a sphere of radius $\frac{1}{\kappa}$ and center $\frac{1}{\kappa}A$. QED

Problem (4.7). Show: A regular surface, every point of which is a planar point, is part of a plane. Hint: Use $\kappa_1 = \kappa_2 = 0$ to obtain $\frac{\partial N}{\partial u} = 0$ & $\frac{\partial N}{\partial v} = 0$.

Remark A right circular cylinder is locally isometric with a plane, but this surface has no umbilics, only parabolic points. Lines of curvature and related notions are extrinsic.

Problem (4.8) Show: A curve $\delta(\alpha(t))$ on a surface for which $\frac{d}{dt} N(\alpha(t))$ is parallel to $\frac{d}{dt}\delta(\alpha(t))$ is a line of curvature. (Converse of Rodrigues.) That is, show

$$\frac{d}{dt} N(\alpha(t)) = \lambda(t) \frac{d}{dt}\delta(\alpha(t))$$

implies $\lambda(t) = \kappa_1(\alpha(t))$ or $\kappa_2(\alpha(t))$. and $\frac{d}{dt}\delta(\alpha(t))$ is the corresponding principal direction.